

The factorized F-matrices for arbitrary $U(1)^{(N-1)}$ integrable vertex models

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Abstract

We discuss the F -matrices associated to the R -matrix of a general N -state vertex model whose statistical configurations encode $N-1$ $U(1)$ symmetries. The factorization condition is shown for arbitrary weights being based only on the unitarity property and the Yang-Baxter relation satisfied by the R -matrix. Focusing on the $N=3$ case we are able to conjecture the structure of some relevant twisted monodromy matrix elements for general weights. We apply this result providing the algebraic expressions of the domain wall partition functions built up in terms of the creation and annihilation monodromy fields. For $N=3$ we also exhibit a R -matrix whose weights lie on a del Pezzo surface and have a rather general structure.

Keywords: N-state Vertex Model, F-Basis, Monodromy Matrix

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1 Introduction

The R -matrix plays a fundamental role in the construction of two-dimensional integrable systems of statistical mechanics. This operator represented here by $R_{ab}(\xi_a, \xi_b)$ acts on the tensor product of two N -dimensional vectors spaces $V_a \otimes V_b$ depending on the complex parameters ξ_a and ξ_b . The R -matrix is required to satisfy the Yang-Baxter equation [1],

$$R_{12}(\xi_1, \xi_2)R_{13}(\xi_1, \xi_3)R_{23}(\xi_2, \xi_3) = R_{23}(\xi_2, \xi_3)R_{13}(\xi_1, \xi_3)R_{12}(\xi_1, \xi_2). \quad (1)$$

The inverse of the R -matrix can be assured by imposing the unitarity condition,

$$R_{12}(\xi_1, \xi_2)R_{21}(\xi_2, \xi_1) = \mathcal{I}_1 \otimes \mathcal{I}_2. \quad (2)$$

where \mathcal{I}_a is the $N \times N$ identity matrix in V_a .

It turns out that the tensor products of the R -matrices called monodromy operators are central objects in the theory of integrable systems [2, 3]. In recent years, it has been realized that such monodromy matrix can be decomposed in a suitable way by means of auxiliary operators that have been denominated F -matrices [4]. This concept was originally introduced for the six-vertex model motivated by the notion of twist deformations of quantum groups [5]. Lets us denote by $R_{1\dots L}^{\{\sigma\}}(\xi_1, \dots, \xi_L)$ the product of R -matrices associated to an arbitrary permutation σ of the symmetry group S_L . The factorization condition for the invertible F -matrices defined by any element $\sigma(1, \dots, L) = \{\sigma(1), \dots, \sigma(L)\} \in S_L$ reads as [4, 6],

$$F_{\sigma(1)\dots\sigma(L)}(\xi_{\sigma(1)}, \dots, \xi_{\sigma(L)})R_{1\dots L}^{\{\sigma\}}(\xi_1, \dots, \xi_L) = F_{1\dots L}(\xi_1, \dots, \xi_L). \quad (3)$$

where the F -matrices $F_{1\dots L}(\xi_1, \dots, \xi_L)$ act on the tensor product spaces $V_1 \otimes \dots \otimes V_L$.

The F -matrices can be used as a natural basis to transform the monodromy matrix in a way that it becomes totally symmetric with respect to a general permutation of the indices $1 \dots L$. This similarity transformation for the six-vertex model permits the development of an alternative approach [6] to deal with the combinatorial problem underlying the general theory of the scalar product of Bethe states [7, 8] and the respective computation of domain wall partition functions [9]. In some respect this method paved the way for further progress on the formulation of the

correlation functions for the spin-1/2 Heisenberg chain [10–12]. It also prompted the search for explicit forms of F -matrices associated to other integrable vertex models such as for certain generalizations of the six-vertex model [13, 14] as well as for multi-state vertex models whose weights are based on the $SL(n|m)$ superalgebra [15–17]. We also remark that a diagrammatic interpretation of the factorization equations for the symmetric six-vertex model has been discussed in [18].

Recently, it has been argued that the existence of the F -matrices for an arbitrary six-vertex model can be pursued without the need of using any explicit weights parameterization [19]. The structure of the F -matrices depends basically on the statistical configurations encoded in the R -matrix and the verification of the factorization condition (3) can be done by using the algebraic weight constraints derived from the Yang-Baxter (1) and unitarity (2) relations. We think that this point of view of considering the formulation of F -matrices should not be particular to the six-vertex model. In this paper we show that this framework can indeed be generalized to tackle integrable N -state vertex models that are invariant by $N - 1$ $U(1)$ symmetries. Recall that for $N = 2$ one obtains the standard asymmetric six-vertex model. We apply the aforementioned construction to the next simplest case $N = 3$, presenting the algebraic expressions of relevant monodromy matrix elements in the F -basis and the corresponding domain wall partition functions.

This paper is organized as follows. In the next Section we define the $U(1)^{(N-1)}$ invariant vertex models and write the algebraic relations (1,2) for the Boltzmann weights. These explicit relations are required to carry out simplifications independent of parameterizations. We motivate our approach by exhibiting a solution of the Yang-Baxter equation for $N = 3$ which contains a number of free parameters. In Section 3 we discuss a procedure to build up the F -matrices for an arbitrary $U(1)^{(N-1)}$ vertex model. It combines, in an effective way, past formulations of the F -matrices for specific weights [15] with a recent construction devised for the six-vertex model [19]. We use the F -basis in Section 4 to provide the expressions of certain relevant monodromy matrix elements for the $N = 3$ vertex model with general weights. In Sections 5 and 6 we apply these results to exhibit the domain wall partition functions associated to products of creation and annihilation fields. Our conclusions are presented in Section 7. In Appendices A-C we summarize

technical details helpful for the understanding of the main text.

2 The $U(1)^{N-1}$ vertex model

Consider a vertex model whose statistical configurations on both horizontal and vertical links of a square $L \times L$ lattice take values on N possible states. As usual the corresponding row-to-row transfer matrix can be written as the trace over an auxiliary space \mathcal{A}_a of the monodromy operator $\mathcal{T}_{a,1\dots L}(\mu)$. This matrix is constructed by the following ordered product of R -matrices,

$$\mathcal{T}_{a,1\dots L}(\mu) = R_{aL}(\mu, \xi_L) R_{a(L-1)}(\mu, \xi_{L-1}) \dots R_{a1}(\mu, \xi_1).$$

From the local Yang-Baxter equation (1) it follows that the monodromy matrix satisfies the following global intertwining relations called Yang-Baxter algebra,

$$R_{ab}(\mu, \nu) \mathcal{T}_{a,1\dots L}(\mu) \mathcal{T}_{b,1\dots L}(\nu) = \mathcal{T}_{b,1\dots L}(\nu) \mathcal{T}_{a,1\dots L}(\mu) R_{ab}(\mu, \nu). \quad (4)$$

In this paper we shall be considering a particular family of N -state vertex models whose statistical configurations are invariant by $N - 1$ $U(1)$ symmetries. We shall denote the local generators of such $U(1)$ symmetries by $S_j^{(z,i)}$, $i = 1, \dots, N - 1$. This means that the corresponding R -matrix is constrained by the commutation relations,

$$[R_{12}(\xi_1, \xi_2), S_1^{(z,i)} \otimes \mathcal{I}_2 + \mathcal{I}_1 \otimes S_2^{(z,i)}] = 0 \quad , \quad i = 1, \dots, N - 1. \quad (5)$$

In terms of the Wely $N \times N$ matrices, $e_j^{(\alpha\beta)} \in V_j$, the expressions for the $N - 1$ azimuthal spin operators $S_j^{(z,i)}$ are,

$$S_j^{(z,i)} = e_j^{(ii)} - e_j^{((i+1)(i+1))} \quad , \quad i = 1, \dots, N - 1. \quad (6)$$

Taking into account the property outlined in Eqs.(5,6) one finds that the R -matrix $R_{12}(\xi_1, \xi_2)$ has $N(2N - 1)$ non-vanishing weights. For $N = 2$, the operator $S_j^{(z,1)}$ reduces to the third component of the spin 1/2 Pauli matrices and the corresponding statistical system is the fully asymmetrical six-vertex model. The possible statistical configurations for general N are given in terms of three distinct classes of weights denoted here by $a_i(\xi_1, \xi_2)$, $b_{ij}(\xi_1, \xi_2)$ and $c_{ij}(\xi_1, \xi_2)$. In Figure 1 we show the respective vertex configurations on the square lattice.

$$a_i(\xi_1, \xi_2) = \xi_1 \begin{array}{c} i \\ | \\ \hline i \\ | \\ \xi_2 \end{array}, \quad b_{ij}(\xi_1, \xi_2) = \xi_1 \begin{array}{c} j \\ | \\ \hline i \\ | \\ \xi_2 \end{array}, \quad c_{ij}(\xi_1, \xi_2) = \xi_1 \begin{array}{c} i \\ | \\ \hline j \\ | \\ \xi_2 \end{array}$$

Figure 1: Elementary configuration of Boltzmann weights.

From Figure 1 we see that the expression of the associated R -matrix in terms of the Weyl matrices is given by,

$$\begin{aligned} R_{12}(\xi_1, \xi_2) &= \sum_{i=1}^N a_i(\xi_1, \xi_2) e_1^{(ii)} \otimes e_2^{(ii)} + \sum_{\substack{i,j=1 \\ i \neq j}}^N b_{ij}(\xi_1, \xi_2) e_1^{(ii)} \otimes e_2^{(jj)} \\ &+ \sum_{\substack{i,j=1 \\ i \neq j}}^N c_{ij}(\xi_1, \xi_2) e_1^{(ij)} \otimes e_2^{(ji)}. \end{aligned} \quad (7)$$

We recall that the integrable vertex model given by Eq.(7) with parameterized weights has been considered in the literature for some time [20,21]. We stress however that the main results of this work will be established without the need of any specific parameterization of the Boltzmann weights. We shall rely solely on the algebraic relations for the weights $a_i(\xi_1, \xi_2)$, $b_{ij}(\xi_1, \xi_2)$ and $c_{ij}(\xi_1, \xi_2)$ coming from the Yang-Baxter and unitary properties. For that reason we need to quote them here explicitly. By substituting the expression for the R -matrix (7) into Eq.(2) we find that the Boltzmann weights are required to satisfy the three distinct types of relations,

$$a_i(\xi_1, \xi_2) a_i(\xi_2, \xi_1) = 1 \quad , \quad i = 1, \dots, N \quad (8)$$

$$b_{ij}(\xi_1, \xi_2) b_{ji}(\xi_2, \xi_1) + c_{ij}(\xi_1, \xi_2) c_{ij}(\xi_2, \xi_1) = 1 \quad , \quad i \neq j = 1, \dots, N \quad (9)$$

$$b_{ij}(\xi_1, \xi_2) c_{ji}(\xi_2, \xi_1) + c_{ij}(\xi_1, \xi_2) b_{ij}(\xi_2, \xi_1) = 0 \quad , \quad i \neq j = 1, \dots, N. \quad (10)$$

The Yang-Baxter equation (1) generates extra constraints on the Boltzmann weights depending now on three independent rapidities. By substituting Eq.(7) into Eq.(1) one finds that the aforementioned functional relations are much more involved. They can however be written in a

compact way by the following expressions,

$$c_{ij}(\xi_1, \xi_2)c_{ji}(\xi_1, \xi_3)c_{ij}(\xi_2, \xi_3) = c_{ji}(\xi_1, \xi_2)c_{ij}(\xi_1, \xi_3)c_{ji}(\xi_2, \xi_3) \quad i \neq j \quad (11)$$

$$b_{ij}(\xi_1, \xi_2)b_{ik}(\xi_1, \xi_3) = b_{ik}(\xi_1, \xi_2)b_{ij}(\xi_1, \xi_3) \quad i \neq j \neq k \quad (12)$$

$$b_{jk}(\xi_1, \xi_3)b_{ik}(\xi_2, \xi_3) = b_{ik}(\xi_1, \xi_3)b_{jk}(\xi_2, \xi_3) \quad i \neq j \neq k \quad (13)$$

$$\begin{aligned} b_{ij}(\xi_1, \xi_2)a_i(\xi_1, \xi_3)c_{ij}(\xi_2, \xi_3) + c_{ji}(\xi_1, \xi_2)c_{ij}(\xi_1, \xi_3)b_{ij}(\xi_2, \xi_3) \\ = a_i(\xi_1, \xi_2)b_{ij}(\xi_1, \xi_3)c_{ij}(\xi_2, \xi_3) \quad i \neq j \end{aligned} \quad (14)$$

$$\begin{aligned} b_{ij}(\xi_1, \xi_2)a_i(\xi_1, \xi_3)c_{ji}(\xi_2, \xi_3) + c_{ij}(\xi_1, \xi_2)c_{ji}(\xi_1, \xi_3)b_{ij}(\xi_2, \xi_3) \\ = a_i(\xi_1, \xi_2)b_{ij}(\xi_1, \xi_3)c_{ji}(\xi_2, \xi_3) \quad i \neq j \end{aligned} \quad (15)$$

$$\begin{aligned} b_{ji}(\xi_1, \xi_2)c_{ij}(\xi_1, \xi_3)b_{ij}(\xi_2, \xi_3) + c_{ij}(\xi_1, \xi_2)a_i(\xi_1, \xi_3)c_{ij}(\xi_2, \xi_3) \\ = a_i(\xi_1, \xi_2)c_{ij}(\xi_1, \xi_3)a_i(\xi_2, \xi_3) \quad i \neq j \end{aligned} \quad (16)$$

$$\begin{aligned} b_{ij}(\xi_1, \xi_2)c_{ij}(\xi_1, \xi_3)b_{ji}(\xi_2, \xi_3) + c_{ij}(\xi_1, \xi_2)a_j(\xi_1, \xi_3)c_{ij}(\xi_2, \xi_3) \\ = a_j(\xi_1, \xi_2)c_{ij}(\xi_1, \xi_3)a_j(\xi_2, \xi_3) \quad i \neq j \end{aligned} \quad (17)$$

$$\begin{aligned} c_{ij}(\xi_1, \xi_2)a_i(\xi_1, \xi_3)b_{ji}(\xi_2, \xi_3) + b_{ji}(\xi_1, \xi_2)c_{ij}(\xi_1, \xi_3)c_{ji}(\xi_2, \xi_3) \\ = c_{ij}(\xi_1, \xi_2)b_{ji}(\xi_1, \xi_3)a_i(\xi_2, \xi_3) \quad i \neq j \end{aligned} \quad (18)$$

$$\begin{aligned} c_{ij}(\xi_1, \xi_2)a_j(\xi_1, \xi_3)b_{ij}(\xi_2, \xi_3) + b_{ij}(\xi_1, \xi_2)c_{ij}(\xi_1, \xi_3)c_{ji}(\xi_2, \xi_3) \\ = c_{ij}(\xi_1, \xi_2)b_{ij}(\xi_1, \xi_3)a_j(\xi_2, \xi_3) \quad i \neq j \end{aligned} \quad (19)$$

$$\begin{aligned} c_{ij}(\xi_1, \xi_2)c_{jk}(\xi_1, \xi_3)b_{ij}(\xi_2, \xi_3) + b_{ij}(\xi_1, \xi_2)c_{ik}(\xi_1, \xi_3)c_{ji}(\xi_2, \xi_3) \\ = c_{ik}(\xi_1, \xi_2)b_{ij}(\xi_1, \xi_3)c_{jk}(\xi_2, \xi_3) \quad i \neq j \neq k \end{aligned} \quad (20)$$

$$\begin{aligned} c_{kj}(\xi_1, \xi_2)c_{ik}(\xi_1, \xi_3)b_{jk}(\xi_2, \xi_3) + b_{jk}(\xi_1, \xi_2)c_{ij}(\xi_1, \xi_3)c_{jk}(\xi_2, \xi_3) \\ = c_{ij}(\xi_1, \xi_2)b_{jk}(\xi_1, \xi_3)c_{ik}(\xi_2, \xi_3) \quad i \neq j \neq k \end{aligned} \quad (21)$$

$$\begin{aligned} b_{ij}(\xi_1, \xi_2)c_{ik}(\xi_1, \xi_3)b_{ji}(\xi_2, \xi_3) + c_{ij}(\xi_1, \xi_2)c_{jk}(\xi_1, \xi_3)c_{ij}(\xi_2, \xi_3) \\ = b_{jk}(\xi_1, \xi_2)c_{ik}(\xi_1, \xi_3)b_{jk}(\xi_2, \xi_3) + c_{jk}(\xi_1, \xi_2)c_{ij}(\xi_1, \xi_3)c_{jk}(\xi_2, \xi_3) \quad i \neq j \neq k \end{aligned} \quad (22)$$

where the indices $i, j, k = 1, \dots, N$.

We would like to conclude this section by remarking that the Yang-Baxter Eqs.(11-22) hide integrable models whose weights structure are indeed rather general. Recall here that the case $N = 2$, the asymmetric six-vertex model, has already been detailed in the literature [1, 22]. Therefore, we shall concentrate on the next simplest system which is the $N = 3$ fifteen-vertex model. The key ingredient to solving integrable models whilst keeping their weights as arbitrary as possible is to uncover the main algebraic varieties constraining the respective Boltzmann weights [1]. In order to tackle this problem for the case $N = 3$ we have adapted a method, first developed in [23], which handles a large number of functional equations associated with three-state vertex models. The technical details of this analysis have been summarized in Appendix A and in what follows we will present only the main results. It turns out that the underlying algebraic variety of one possible solution is a homogeneous hypersurface given by the equation,

$$\left(\frac{\Delta_1 \Delta_2 - 1}{\Delta_1^2} \right) a(\xi_i)^2 b(\xi_i) - \Delta_2 a(\xi_i) b(\xi_i) \bar{b}(\xi_i) + b(\xi_i) \bar{b}(\xi_i)^2 - \bar{b}(\xi_i) c(\xi_i) \bar{c}(\xi_i) = 0 \quad (23)$$

where Δ_1, Δ_2 are free constants while $a(\xi_i)$, $b(\xi_i)$, $\bar{b}(\xi_i)$, $c(\xi_i)$ and $\bar{c}(\xi_i)$ are arbitrary variables depending on the spectral parameters.

One possible way to parameterize the hypersurface (23) is first to consider its intersection with the hyperplane $c(\xi_i) - \bar{c}(\xi_i) = 0$. As a result we obtain an algebraic variety in the class of the cubic del Pezzo surfaces which can be parameterized in terms of rational functions [24]. Following an algorithm devised in [25] we conclude the rational map is attained by just solving Eq.(23) for the linear variable $b(\xi_i)$ in terms of the remaining parameters $a(\xi_i)$, $\bar{b}(\xi_i)$ and $c(\xi_i)$. Clearly, the same type of procedure also works for the general manifold (23) and therefore the R -matrix depends at least on the three free variables $a(\xi_i)$, $\bar{b}(\xi_i)$ and $c(\xi_i)$. Taking into account the results of Appendix A we find that the R -matrix elements are given by,

$$\frac{a_1(\xi_1, \xi_2)}{c_{12}(\xi_1, \xi_2)} = \frac{c(\xi_2)}{c(\xi_1)} \frac{[(\Delta_1 \Delta_2 - 1)a(\xi_1)a(\xi_2) - \Delta_1^2 \bar{b}(\xi_2)(\Delta_2 a(\xi_1) - \bar{b}(\xi_1))]}{[a(\xi_2) - \Delta_1 \bar{b}(\xi_2)] [(\Delta_1 \Delta_2 - 1)a(\xi_2) - \Delta_1 \bar{b}(\xi_2)]} \quad (24)$$

$$\begin{aligned} \frac{b_{12}(\xi_1, \xi_2)}{c_{12}(\xi_1, \xi_2)} &= \Delta_1^2 (-1 + \Delta_1 \Delta_2) c(\xi_2) \bar{c}(\xi_1) [a(\xi_2) \bar{b}(\xi_1) - a(\xi_1) \bar{b}(\xi_2)] \\ &\times \frac{[-a(\xi_2) + \Delta_1 \bar{b}(\xi_2)]^{-1} [(\Delta_1 \Delta_2 - 1)a(\xi_2) - \Delta_1 \bar{b}(\xi_2)]^{-1}}{[-a(\xi_1) + \Delta_1 \bar{b}(\xi_1)] [(\Delta_1 \Delta_2 - 1)a(\xi_1) - \Delta_1 \bar{b}(\xi_1)]} \end{aligned} \quad (25)$$

$$\frac{b_{13}(\xi_1, \xi_2)}{c_{12}(\xi_1, \xi_2)} = \frac{\Delta_1^2(\Delta_1\Delta_2 - 1)c(\xi_2)h_1(\xi_1)h_2(\xi_1)}{\delta_2 c(\xi_1)\bar{c}(\xi_1)} \frac{[a(\xi_2)\bar{b}(\xi_1) - a(\xi_1)\bar{b}(\xi_2)]}{[a(\xi_2) - \Delta_1\bar{b}(\xi_2)]} \times \\ [(\Delta_1\Delta_2 - 1)a(\xi_1) - \Delta_1\bar{b}(\xi_1)]^{-1} [(\Delta_1\Delta_2 - 1)a(\xi_2) - \Delta_1\bar{b}(\xi_2)]^{-1} \quad (26)$$

$$\frac{b_{21}(\xi_1, \xi_2)}{c_{12}(\xi_1, \xi_2)} = \frac{a(\xi_2)\bar{b}(\xi_1) - a(\xi_1)\bar{b}(\xi_2)}{c(\xi_1)\bar{c}(\xi_2)} \quad (27)$$

$$\frac{a_2(\xi_1, \xi_2)}{c_{12}(\xi_1, \xi_2)} = \frac{\bar{c}(\xi_1)}{\bar{c}(\xi_2)} \frac{[(\Delta_1\Delta_2 - 1)a(\xi_1)a(\xi_2) - \Delta_1^2\bar{b}(\xi_2)(\Delta_2a(\xi_1) - \bar{b}(\xi_1))]}{[a(\xi_1) - \Delta_1\bar{b}(\xi_1)] [-a(\xi_1) + \Delta_1\Delta_2a(\xi_1) - \Delta_1\bar{b}(\xi_1)]} \quad (28)$$

$$\frac{b_{23}(\xi_1, \xi_2)}{c_{12}(\xi_1, \xi_2)} = \frac{(\Delta_1\Delta_2 - 1)h_1(\xi_1)h_2(\xi_1)}{\delta_1 c(\xi_1)\bar{c}(\xi_1)\bar{c}(\xi_2)} \frac{[a(\xi_2)\bar{b}(\xi_1) - a(\xi_1)\bar{b}(\xi_2)]}{[(\Delta_1\Delta_2 - 1)a(\xi_1) - \Delta_1\bar{b}(\xi_1)]} \quad (29)$$

$$\frac{c_{13}(\xi_1, \xi_2)}{c_{12}(\xi_1, \xi_2)} = \frac{c(\xi_2)\bar{c}(\xi_2)h_1(\xi_1)}{c(\xi_1)\bar{c}(\xi_1)h_1(\xi_2)} \frac{[a(\xi_1) - \Delta_1\bar{b}(\xi_1)]}{[a(\xi_2) - \Delta_1\bar{b}(\xi_2)]} \quad (30)$$

$$\frac{c_{21}(\xi_1, \xi_2)}{c_{12}(\xi_1, \xi_2)} = \frac{c(\xi_2)\bar{c}(\xi_1)}{c(\xi_1)\bar{c}(\xi_2)}, \quad \frac{c_{23}(\xi_1, \xi_2)}{c_{12}(\xi_1, \xi_2)} = \frac{c(\xi_2)h_1(\xi_1)}{c(\xi_1)h_1(\xi_2)}, \quad \frac{c_{31}(\xi_1, \xi_2)}{c_{12}(\xi_1, \xi_2)} = \frac{c(\xi_2)h_2(\xi_1)}{c(\xi_1)h_2(\xi_2)} \quad (31)$$

$$\frac{b_{31}(\xi_1, \xi_2)}{c_{12}(\xi_1, \xi_2)} = \frac{\delta_2 c(\xi_2)\bar{c}(\xi_2)}{h_1(\xi_2)c(\xi_1)h_2(\xi_2)} \frac{[a(\xi_2)\bar{b}(\xi_1) - a(\xi_1)\bar{b}(\xi_2)]}{[a(\xi_2) - \Delta_1\bar{b}(\xi_2)]} \quad (32)$$

$$\frac{c_{32}(\xi_1, \xi_2)}{c_{12}(\xi_1, \xi_2)} = \frac{h_2(\xi_1)}{h_2(\xi_2)} \frac{[(\Delta_1\Delta_2 - 1)a(\xi_2) - \Delta_1\bar{b}(\xi_2)]}{[(\Delta_1\Delta_2 - 1)a(\xi_1) - \Delta_1\bar{b}(\xi_1)]} \quad (33)$$

$$\frac{b_{32}(\xi_1, \xi_2)}{c_{12}(\xi_1, \xi_2)} = \frac{\Delta_1^2\delta_1 c(\xi_2)\bar{c}(\xi_1)\bar{c}(\xi_2)}{h_1(\xi_2)h_2(\xi_2)} \frac{[a(\xi_2)\bar{b}(\xi_1) - a(\xi_1)\bar{b}(\xi_2)]}{[(\Delta_1\Delta_2 - 1)a(\xi_1) - \Delta_1\bar{b}(\xi_1)] [a(\xi_1) - \Delta_1\bar{b}(\xi_1)] [a(\xi_2) - \Delta_1\bar{b}(\xi_2)]} \quad (34)$$

$$\frac{a_3(\xi_1, \xi_2)}{c_{12}(\xi_1, \xi_2)} = \frac{h_1(\xi_1)c(\xi_2)\bar{c}(\xi_2)h_2(\xi_1)}{h_1(\xi_2)c(\xi_1)\bar{c}(\xi_1)h_2(\xi_2)} \frac{[(\Delta_1\Delta_2 - 1)a(\xi_1)a(\xi_2) - \Delta_1^2\bar{b}(\xi_2)(\Delta_2a(\xi_1) - \bar{b}(\xi_1))]}{[(\Delta_1\Delta_2 - 1)a(\xi_1) - \Delta_1\bar{b}(\xi_1)] [a(\xi_2) - \Delta_1\bar{b}(\xi_2)]} \quad (35)$$

where $c_{12}(\xi_1, \xi_2)$ is an overall normalization, δ_1 and δ_2 are extra free constants while $h_1(\xi_i)$ and $h_2(\xi_i)$ are free variables. The latter freedom is related to the fact that the Yang-Baxter equation is preserved under local transformations associated to the two $U(1)$ symmetries.

The structure of the R -matrix is certainly more general than that of the standard $U(1) \otimes U(1)$ invariant vertex models containing only one free spectral parameter [20, 21]. This rather particular parameterization for the completely free variables $a(\xi_i)$, $\bar{b}(\xi_i)$ and $c(\xi_i)$ is discussed at the end of Appendix A. This fact highlights the importance to search for explicit results in integrable models that are independent of specific parameterization of the Boltzmann weights.

3 Factorized F-matrices

We begin by providing some basic definitions and notation that are going to be used in the text. We first recall that an element σ of the symmetric group S_L can be generated in terms of adjacent permutations. In other words one can write,

$$\sigma = \sigma_{\alpha_p(\alpha_p+1)} \cdots \sigma_{\alpha_2(\alpha_2+1)} \sigma_{\alpha_1(\alpha_1+1)}, \quad (36)$$

where $\sigma_{\alpha(\alpha+1)}$ denotes the permutation of the indices α and $\alpha+1$, i.e. $\sigma_{\alpha(\alpha+1)}(1, \dots, \alpha, \alpha+1, \dots, L) = (1, \dots, \alpha+1, \alpha, \dots, L)$. We shall refer to Eq. (36) as the minimum decomposition of σ .

In order to avoid cumbersome notation by the presence of the inhomogeneities ξ_i in the F -matrices we shall omit them when this does not generates confusion. In general, we shall identify a given general element $X_{a,\sigma(1),\dots,\sigma(L)}(\mu|\xi_{\sigma(1)},\dots,\xi_{\sigma(L)}) \in \text{End}(\mathcal{A}_a \otimes V_{\sigma(1)} \otimes \cdots \otimes V_{\sigma(L)})$ to $X_{a,\sigma(1\dots L)}(\mu)$. Taking this notation into account the factorization condition (3) is rewritten as,

$$F_{\sigma(1\dots L)} R_{1\dots L}^{\{\sigma\}} = F_{1\dots L}. \quad (37)$$

The tensor product of R -matrices $R_{1\dots L}^{\{\sigma\}}$ entering Eq.(37) is defined in terms of product of auxiliary operators by the expression,

$$R_{1\dots L}^{\{\sigma\}} = P_{1\dots L}^{\{\sigma\}} \hat{R}_{1\dots L}^{\{\sigma^{-1}\}}. \quad (38)$$

Through considering the minimum decomposition of σ in terms of adjacent permutations, the auxiliary operators $P_{1\dots L}^{\{\sigma\}}$ and $\hat{R}_{1\dots L}^{\{\sigma^{-1}\}}$ can be written in the following way,

$$\begin{aligned} P_{1\dots L}^{\{\sigma\}} &= P_{1\dots L}^{\{\sigma_{\alpha_p(\alpha_p+1)}\}} \cdots P_{1\dots L}^{\{\sigma_{\alpha_2(\alpha_2+1)}\}} P_{1\dots L}^{\{\sigma_{\alpha_1(\alpha_1+1)}\}} \\ \hat{R}_{1\dots L}^{\{\sigma^{-1}\}} &= \hat{R}_{1\dots L}^{\{\sigma_{\alpha_1(\alpha_1+1)}\}} \hat{R}_{1\dots L}^{\{\sigma_{\alpha_2(\alpha_2+1)}\}} \cdots \hat{R}_{1\dots L}^{\{\sigma_{\alpha_p(\alpha_p+1)}\}}, \end{aligned} \quad (39)$$

where $P_{1\dots L}^{\{\sigma_{\alpha(\alpha+1)}\}} = P_{\alpha(\alpha+1)}$ is the standard permutator while $\hat{R}_{1\dots L}^{\{\sigma_{\alpha(\alpha+1)}\}} = P_{\alpha(\alpha+1)} R_{\alpha(\alpha+1)}$.

We now list a number of properties that are necessary in our analysis of the factorization condition (37). We start by mentioning that for any given operator $X_{1\dots L} \in \text{End}(V_1 \otimes \cdots \otimes V_L)$ we have the following useful relation,

$$X_{\sigma(1\dots L)} = P_{1\dots L}^{\{\sigma\}} X_{1\dots L} P_{1\dots L}^{\{\sigma^{-1}\}}. \quad (40)$$

Next, given the decomposition laws of the auxiliary operators (39) one can show that for general $\{\sigma, \tau\} \in S_L$ the tensor product of permuted R -matrices obeys the following decomposition identity,

$$R_{1\dots L}^{\{\sigma\tau\}} = R_{\sigma(1\dots L)}^{\{\tau\}} R_{1\dots L}^{\{\sigma\}}. \quad (41)$$

In order to verify Eq.(41) we consider the expression $R_{1\dots L}^{\{\sigma\tau\}}$ and perform the following operations,

$$\begin{aligned} \underbrace{R_{1\dots L}^{\{\sigma\tau\}}}_{\text{apply Eq.(38)}} &= P_{1\dots L}^{\{\sigma\tau\}} \hat{R}_{1\dots L}^{\{(\sigma\tau)^{-1}\}} = P_{1\dots L}^{\{\sigma\}} \underbrace{P_{1\dots L}^{\{\tau\}} \hat{R}_{1\dots L}^{\{\tau^{-1}\}} \hat{R}_{1\dots L}^{\{\sigma^{-1}\}}}_{\text{apply Eq.(38)}} \\ &= \underbrace{P_{1\dots L}^{\{\sigma\}} R_{1\dots L}^{\{\tau\}} P_{1\dots L}^{\{\sigma^{-1}\}}}_{\text{apply Eq.(40)}} \underbrace{P_{1\dots L}^{\{\sigma\}} \hat{R}_{1\dots L}^{\{\sigma^{-1}\}}}_{\text{apply Eq.(38)}} = R_{\sigma(1\dots L)}^{\{\tau\}} R_{1\dots L}^{\{\sigma\}}. \end{aligned} \quad (42)$$

We end by mentioning another standard identity regarding the action of the operator $R_{1\dots L}^{\{\sigma\}}$ on the monodromy matrix, namely

$$R_{1\dots L}^{\{\sigma\}} \mathcal{T}_{a,1\dots L}(\mu) = \mathcal{T}_{a,\sigma(1\dots L)}(\mu) R_{1\dots L}^{\{\sigma\}}. \quad (43)$$

3.1 The $L = 2$ example

We shall start the explicit construction of the F -matrices for arbitrary weights $a_i(\xi_1, \xi_2)$, $b_{ij}(\xi_1, \xi_2)$ and $c_{ij}(\xi_1, \xi_2)$. Besides the factorization relation (37) we also require that the F -matrices are lower triangular and invertible [4]. In this sense it is instructive to begin with the simplest case $L = 2$. In this situation the only non-trivial permutation is the adjacent permutation σ_{12} and Eq.(37) becomes,

$$F_{21} R_{12} = F_{12}. \quad (44)$$

We shall now explicitly show that the solution the expression (44) is given by,

$$F_{12} = \mathcal{N}_{12} \mathcal{F}_{12} \quad (45)$$

where \mathcal{N}_{12} is the diagonal matrix,

$$\mathcal{N}_{12} = \mathcal{I}_1 \otimes \mathcal{I}_2 - \sum_{i=1}^N e_1^{(ii)} \otimes e_2^{(ii)} + \sum_{i=1}^N \sqrt{a_i(\xi_1, \xi_2)} e_1^{(ii)} \otimes e_2^{(ii)}, \quad (46)$$

and \mathcal{F}_{12} is the given by the following operator,

$$\mathcal{F}_{12} = \sum_{1 \leq \alpha_1 \leq \alpha_2 \leq N} e_1^{(\alpha_1 \alpha_1)} \otimes e_2^{(\alpha_2 \alpha_2)} \mathcal{I}_1 \otimes \mathcal{I}_2 + \sum_{1 \leq \alpha_2 < \alpha_1 \leq N} e_1^{(\alpha_1 \alpha_1)} \otimes e_2^{(\alpha_2 \alpha_2)} R_{12}. \quad (47)$$

Substituting Eq.(45) into the factorization condition and applying Eq.(8) we obtain,

$$\mathcal{F}_{21} R_{12} = \mathcal{R}_{12} \mathcal{F}_{12}, \quad (48)$$

where the twisted R -matrix, \mathcal{R}_{12} , is the diagonal matrix,

$$\mathcal{R}_{12} = \mathcal{I}_1 \otimes \mathcal{I}_2 - \sum_{i=1}^N e_1^{(ii)} \otimes e_2^{(ii)} + \sum_{i=1}^N a_i(\xi_1, \xi_2) e_1^{(ii)} \otimes e_2^{(ii)}. \quad (49)$$

Hence, applying unitarity (2) the $L = 2$ factorization condition becomes,

$$\begin{aligned} & \sum_{1 \leq \alpha_2 \leq \alpha_1 \leq N} e_1^{(\alpha_1 \alpha_1)} \otimes e_2^{(\alpha_2 \alpha_2)} R_{12} + \sum_{1 \leq \alpha_1 < \alpha_2 \leq N} e_1^{(\alpha_1 \alpha_1)} \otimes e_2^{(\alpha_2 \alpha_2)} \mathcal{I}_1 \otimes \mathcal{I}_2 \\ &= \sum_{1 \leq \alpha_1 \leq \alpha_2 \leq N} \mathcal{R}_{12} e_1^{(\alpha_1 \alpha_1)} \otimes e_2^{(\alpha_2 \alpha_2)} + \sum_{1 \leq \alpha_2 < \alpha_1 \leq N} \mathcal{R}_{12} e_1^{(\alpha_1 \alpha_1)} \otimes e_2^{(\alpha_2 \alpha_2)} R_{12}. \end{aligned} \quad (50)$$

Using the following relation regarding the action of the \mathcal{R} -matrix on elementary matrices,

$$\mathcal{R}_{12} e_1^{(ii)} \otimes e_2^{(jj)} = \begin{cases} a_i(\xi_1, \xi_2) e_1^{(ii)} \otimes e_2^{(jj)} & \text{for } i = j \\ e_1^{(ii)} \otimes e_2^{(jj)} & \text{for } i \neq j \end{cases}, \quad (51)$$

we can immediately see that the $\alpha_1 \neq \alpha_2$ components of the summations in Eq.(50) cancel. Hence we are left with the expression,

$$\sum_{\alpha=1}^N e_1^{(\alpha \alpha)} \otimes e_2^{(\alpha \alpha)} R_{12} = \sum_{\alpha=1}^N \mathcal{R}_{12} e_1^{(\alpha \alpha)} \otimes e_2^{(\alpha \alpha)}, \quad (52)$$

which is true by inspection.

We end by commenting that the form of \mathcal{F}_{12} coincides exactly with the solution originally proposed in [15] for the rational $SU(N)$ vertex models. This observation provides us a hint on how to proceed for arbitrary L .

3.2 The general L case

For general L we provide the following ansatz for the form of $F_{1\dots L}$,

$$F_{1\dots L} = \mathcal{N}_{1\dots L} \mathcal{F}_{1\dots L}, \quad (53)$$

where the definition of $\mathcal{N}_{1\dots L}$ is given by the product of *partial* \mathcal{N} -matrices,

$$\mathcal{N}_{1\dots L} = \mathcal{N}_{2\dots L} \mathcal{N}_{1,2\dots L} = \mathcal{N}_{(L-1)L} \mathcal{N}_{L-2,(L-1)L} \dots \mathcal{N}_{1,2\dots L}, \quad (54)$$

such that the partial \mathcal{N} -matrices $\mathcal{N}_{i,(i+1)\dots L}$, are given by,

$$\mathcal{N}_{i,(i+1)\dots L} = \mathcal{N}_{iL} \mathcal{N}_{i(L-1)} \dots \mathcal{N}_{i(i+1)}. \quad (55)$$

As with the $L = 2$ case, the terms $\mathcal{F}_{1\dots L}$ coincide with the form of the solution given in [15]:

$$\mathcal{F}_{1\dots L} = \sum_{\sigma \in S_L} \sum_{1 \leq \alpha_{\sigma(1)} \dots \alpha_{\sigma(L)} \leq N}^* \bigotimes_{i=1}^L e_i^{(\alpha_i \alpha_i)} R_{1\dots L}^{\{\sigma\}}, \quad (56)$$

where the symbol $*$ in the sum (56) of ordered indices is to be over all non decreasing sequences of the indices $\alpha_{\sigma(i)}$. The indices $\alpha_{\sigma(i)}$ satisfy one of the two inequalities for each pair of neighboring indices:

$$\begin{aligned} \alpha_{\sigma(i)} &\leq \alpha_{\sigma(i+1)} & \text{if } \sigma(i) < \sigma(i+1) \\ \alpha_{\sigma(i)} &< \alpha_{\sigma(i+1)} & \text{if } \sigma(i) > \sigma(i+1). \end{aligned} \quad (57)$$

The specific choice on the second part of the sum (56) and the form of the R -matrix ensure that $\mathcal{F}_{1\dots L}$ and $F_{1\dots L}$ are lower-triangular (see [15–17] for details). Additionally, since each diagonal entry is non-zero, the inverse $F_{1\dots L}^{-1}$ is assured to exist. Note also that the tensor product term in Eq.(56) is invariant if we apply the permutation to each index i ,

$$\bigotimes_{i=1}^L e_i^{(\alpha_i \alpha_i)} = \bigotimes_{i=1}^L e_{\sigma(i)}^{(\alpha_{\sigma(i)} \alpha_{\sigma(i)})},$$

and hence an equivalent expression for the \mathcal{F} -matrix is,

$$\mathcal{F}_{1\dots L} = \sum_{\sigma \in S_L} \sum_{1 \leq \alpha_{\sigma(1)} \dots \alpha_{\sigma(L)} \leq N}^* \bigotimes_{i=1}^L e_{\sigma(i)}^{(\alpha_{\sigma(i)} \alpha_{\sigma(i)})} R_{1\dots L}^{\{\sigma\}}. \quad (58)$$

We now are left with the task of verifying the factorization condition (37). This involves a sequence of steps that we shall now detail.

3.2.1 Recasting the factorization condition

In order to show the validity of the factorization condition (37) for all $\sigma \in S_L$ we proceed much like [19] and take advantage of the decomposition of $F_{1\dots L}$ present in Eq.(53). In doing so we recast Eq.(37) as an equation involving $\mathcal{F}_{1\dots L}$ and the twisted R -matrix. We then adapt the procedure first devised in [15–17] to tackle the problem. To this end we substitute Eq.(53) into Eq.(37) to obtain,

$$\mathcal{F}_{\sigma(1\dots L)} R_{1\dots L}^{\{\sigma\}} = \mathcal{N}_{\sigma(1\dots L)}^{-1} \mathcal{N}_{1\dots L} \mathcal{F}_{1\dots L} \quad \text{for all } \sigma \in S_L. \quad (59)$$

Recall that all elements of S_L possess a minimal decomposition in terms of adjacent permutations (36). For the expression $\mathcal{N}_{\sigma(1\dots L)}^{-1} \mathcal{N}_{1\dots L}$ we offer the following result.

Proposition 1.

$$\mathcal{N}_{\sigma(1\dots L)}^{-1} \mathcal{N}_{1\dots L} = \mathcal{R}_{1\dots L}^{\{\sigma\}} \quad \text{for all } \sigma \in S_L, \quad (60)$$

where $\mathcal{R}_{1\dots L}^{\{\sigma\}}$ follows the same decomposition rules as $R_{1\dots L}^{\{\sigma\}}$,

$$\begin{aligned} \mathcal{R}_{1\dots L}^{\{\sigma\}} &= P_{1\dots L}^{\{\sigma\}} \hat{\mathcal{R}}_{1\dots L}^{\{\sigma^{-1}\}} \\ \text{where } \hat{\mathcal{R}}_{1\dots L}^{\{\sigma^{-1}\}} &= \hat{\mathcal{R}}_{1\dots L}^{\{\sigma_{\alpha_1(\alpha_1+1)}\}} \hat{\mathcal{R}}_{1\dots L}^{\{\sigma_{\alpha_2(\alpha_2+1)}\}} \dots \hat{\mathcal{R}}_{1\dots L}^{\{\sigma_{\alpha_p(\alpha_p+1)}\}} \\ \text{and } \hat{\mathcal{R}}_{1\dots L}^{\{\sigma_{\alpha(\alpha+1)}\}} &= P_{\alpha(\alpha+1)} \mathcal{R}_{\alpha(\alpha+1)}, \end{aligned} \quad (61)$$

and \mathcal{R}_{12} is explicitly given by Eq.(49).

Because the corresponding auxiliary operators $\hat{\mathcal{R}}_{1\dots L}^{\{\sigma\}}$ provide a valid representation for S_L , the above result can be verified by showing that Eq.(60) holds for only two permutations: the adjacent permutation σ_{12} and the cyclic permutation $\sigma_c = \sigma_{12}\sigma_{23}\dots\sigma_{(L-1)L}$. For more details refer to Appendix C of [19].

Hence we recast the factorization condition in the following form,

$$\mathcal{F}_{\sigma(1\dots L)} R_{1\dots L}^{\{\sigma\}} = \mathcal{R}_{1\dots L}^{\{\sigma\}} \mathcal{F}_{1\dots L} \quad \text{for all } \sigma \in S_L. \quad (62)$$

We impose that the \mathcal{R} -matrices follow the same left-handed and right-handed convention as the R -matrices,

$$\mathcal{R}_{1,2\dots,N} = \mathcal{R}_{1N} \mathcal{R}_{1(N-1)} \dots \mathcal{R}_{12}, \quad \text{and} \quad \mathcal{R}_{1\dots,N-1,N} = \mathcal{R}_{1N} \mathcal{R}_{2N} \dots \mathcal{R}_{(N-1)N}. \quad (63)$$

Following the above convention the \mathcal{R} -matrices obey the same global unitarity condition as the R -matrices,

$$\mathcal{R}_{1,2\dots L}\mathcal{R}_{2\dots L,1} = \mathcal{I}_1 \otimes \dots \otimes \mathcal{I}_L, \quad (64)$$

and being diagonal the \mathcal{R} -matrices commute amongst themselves.

3.3 Verification of the factorization property

We begin by applying Eq.(40) to Eq.(62) to obtain,

$$\hat{R}_{1\dots L}^{\{\sigma^{-1}\}} = \mathcal{F}_{1\dots L}^{-1} \hat{\mathcal{R}}_{1\dots L}^{\{\sigma^{-1}\}} \mathcal{F}_{1\dots L}. \quad (65)$$

Since both $\hat{R}_{1\dots L}^{\{\sigma\}}$ and $\hat{\mathcal{R}}_{1\dots L}^{\{\sigma\}}$ provide valid representations of S_L , we remark that Eq.(65) is in a form that one can readily decompose the permutation σ . To illustrate this consider the permutation $\sigma = \sigma_1\sigma_2$ on the left-hand side and right-hand side of Eq.(65) respectively,

$$\begin{aligned} \hat{R}_{1\dots L}^{\{(\sigma_1\sigma_2)^{-1}\}} &= \hat{R}_{1\dots L}^{\{\sigma_2^{-1}\}} \hat{R}_{1\dots L}^{\{\sigma_1^{-1}\}} \\ \mathcal{F}_{1\dots L}^{-1} \hat{\mathcal{R}}_{1\dots L}^{\{(\sigma_1\sigma_2)^{-1}\}} \mathcal{F}_{1\dots L} &= \mathcal{F}_{1\dots L}^{-1} \hat{\mathcal{R}}_{1\dots L}^{\{\sigma_2^{-1}\}} \mathcal{F}_{1\dots L} \mathcal{F}_{1\dots L}^{-1} \hat{\mathcal{R}}_{1\dots L}^{\{\sigma_1^{-1}\}} \mathcal{F}_{1\dots L} \end{aligned} \quad (66)$$

where $\{\sigma_1, \sigma_2\} \in S_L$.

Since S_L can be constructed entirely from the adjacent permutations $\sigma_{j(j+1)}$, $j = (1, \dots, L-1)$, we need only verify Eq.(62) for the adjacent permutation to guarantee its validity for all S_L . To this end we substitute $\sigma = \sigma_{j(j+1)}$ into the factorization condition (62) obtaining,

$$\mathcal{F}_{\sigma_{j(j+1)}(1\dots L)} R_{j(j+1)} - \mathcal{R}_{j(j+1)} \mathcal{F}_{1\dots L} = 0, \quad (67)$$

where,

$$\begin{aligned} \mathcal{F}_{\sigma_{j(j+1)}(1\dots L)} R_{j(j+1)} &= \sum_{\sigma \in S_L} \sum_{1 \leq \alpha_{\sigma(1)} \dots \alpha_{\sigma(L)} \leq N}^* \bigotimes_{i=1}^L e_{\sigma_{j(j+1)}\sigma(i)}^{(\alpha_{\sigma(i)} \alpha_{\sigma(i)})} \overbrace{R_{\sigma_{j(j+1)}(1\dots L)}^{\{\sigma\}} R_{1\dots L}^{\sigma_{j(j+1)}}}^{\text{apply Eq.(41)}} \\ &= \sum_{\sigma \in S_L} \sum_{1 \leq \alpha_{\sigma(1)} \dots \alpha_{\sigma(L)} \leq N}^* \bigotimes_{i=1}^L e_{\sigma_{j(j+1)}\sigma(i)}^{(\alpha_{\sigma(i)} \alpha_{\sigma(i)})} R_{1\dots L}^{\{\sigma_{j(j+1)}\sigma\}}. \end{aligned} \quad (68)$$

We now perform the change of variables $\alpha_{\sigma(i)} \rightarrow \alpha_{\sigma_{j(j+1)}\sigma(i)}$ to the summation indices $\alpha_{\sigma(i)}$. In doing so Eq.(68) becomes,

$$\begin{aligned}\mathcal{F}_{\sigma_{j(j+1)}(1\dots L)}R_{j(j+1)} &= \sum_{\sigma \in S_L} \sum_{1 \leq \alpha_{\sigma_{j(j+1)}\sigma(1)} \dots \alpha_{\sigma_{j(j+1)}\sigma(L)} \leq N}^{**} \bigotimes_{i=1}^L e_{\sigma_{j(j+1)}\sigma(i)}^{(\alpha_{\sigma_{j(j+1)}\sigma(i)} \alpha_{\sigma_{j(j+1)}\sigma(i)})} R_{1\dots L}^{\{\sigma_{j(j+1)}\sigma\}} \\ &= \sum_{\tau \in S_L} \sum_{1 \leq \alpha_{\tau(1)} \dots \alpha_{\tau(L)} \leq N}^{**} \bigotimes_{i=1}^L e_{\tau(i)}^{(\alpha_{\tau(i)} \alpha_{\tau(i)})} R_{1\dots L}^{\{\tau\}},\end{aligned}\tag{69}$$

where we have applied the relabeling $\sigma_{j(j+1)}\sigma = \tau$ of the elements of S_L .

The summation $\sum_{1 \leq \alpha_{\tau(1)} \dots \alpha_{\tau(L)} \leq N}^{**}$ in Eq.(69) for $\mathcal{F}_{\sigma_{j(j+1)}(1\dots L)}R_{j(j+1)}$ is deceptively similar to $\sum_{1 \leq \alpha_{\sigma(1)} \dots \alpha_{\sigma(L)} \leq N}^*$ in Eq.(58) for $\mathcal{F}_{1\dots L}$, in that its ordered indices are to be summed over all non decreasing sequences of the indices $\alpha_{\tau(i)}$. However the indices $\alpha_{\tau(i)}$ satisfy one of the two inequalities for each pair of neighboring indices:

$$\begin{aligned}\alpha_{\tau(i)} &\leq \alpha_{\tau(i+1)} & \text{if } \sigma_{j(j+1)}\tau(i) < \sigma_{j(j+1)}\tau(i+1) \\ \alpha_{\tau(i)} &< \alpha_{\tau(i+1)} & \text{if } \sigma_{j(j+1)}\tau(i) > \sigma_{j(j+1)}\tau(i+1)\end{aligned}\tag{70}$$

Comparing (57) with (70), one can see that the only difference between them is the adjacent permutation $\sigma_{j(j+1)}$ factor in the *if* conditions. For any given $\tau \in S_L$, we focus on two integers, k and l , where $\tau(k) = j$ and $\tau(l) = j+1$. Using these two integers we examine how the elementary transposition $\sigma_{j(j+1)}$ will affect the inequalities in (70) compared to (57). There are only two relevant cases, given by $|k-l| > 1$ and $|k-l| = 1$. When $|k-l| > 1$ then the adjacent permutation $\sigma_{j(j+1)}$ does not affect the sequence of indices α_{τ} at all, meaning that (57) and (70) are the same. To explicitly show that (57) and (70) are the same in this case we concentrate on the following four tuples,

$$(\alpha_{\tau(k-1)}, \alpha_{\tau(k)}) \quad (\alpha_{\tau(k)}, \alpha_{\tau(k+1)}) \quad (\alpha_{\tau(l-1)}, \alpha_{\tau(l)}) \quad (\alpha_{\tau(l)}, \alpha_{\tau(l+1)}),$$

which are affected by the permutation $\sigma_{j(j+1)}$.

Focusing on the tuple $(\alpha_{\tau(k-1)}, \alpha_{\tau(k)})$, there are two possible cases for the values of $\tau(k-1)$ and $\tau(k)$ given by,

$$\sigma_{j(j+1)}\tau(k-1) > \sigma_{j(j+1)}\tau(k) \quad \text{or} \quad \sigma_{j(j+1)}\tau(k-1) < \sigma_{j(j+1)}\tau(k)\tag{71}$$

Since $|k-l| > 1$ we remark that for both cases given in Eq.(71):

- $\tau(k-1)$ is invariant under the permutation $\sigma_{j(j+1)}$.
- $\sigma_{j(j+1)}\tau(k) = \sigma_{j(j+1)}(j) = j+1$ - changing $\tau(k)$ to $\tau(k)+1$ is not going to affect the values of the inequalities.

An equivalently elementary analysis can be performed for the three remaining tuples. Hence with these observations, we are assured that the type of the inequality is unchanged by the addition of the permutation $\sigma_{j(j+1)}$ - meaning that (57) and (70) lead to the same results when $|k-l| > 1$.

We now concentrate on the case $|k-l| = 1$. In this situation the permutation $\sigma_{j(j+1)}$ does affect the inequality, leading to a difference between (57) and (70). To see this consider the case $\tau(k) = j$ and $\tau(k+1) = j+1$, where

$$\begin{aligned}\tau(k) &< \tau(k+1) && \text{and} \\ \sigma_{j(j+1)}\tau(k) &> \sigma_{j(j+1)}\tau(k+1).\end{aligned}$$

Thus, what is usually a “<” inequality in (57) changes to a “>” inequality in (70). Hence the summation in Eq.(56),

$$\sum_{\dots\alpha_{\tau(k)}\alpha_{\tau(k+1)}\dots}^* = \sum_{\dots\alpha_j\leq\alpha_{j+1}\dots},$$

changes when we consider the summation in Eq.(69),

$$\sum_{\dots\alpha_{\tau(k)}\alpha_{\tau(k+1)}\dots}^{**} = \sum_{\dots\alpha_j<\alpha_{j+1}\dots}.$$

Equivalently, consider the case $\tau(k) = j+1$ and $\tau(k+1) = j$, where

$$\begin{aligned}\tau(k) &> \tau(k+1) && \text{and} \\ \sigma_{j(j+1)}\tau(k) &< \sigma_{j(j+1)}\tau(k+1).\end{aligned}$$

Thus, what is usually a “>” inequality in (57) changes to a “<” inequality in (70). Hence the summation in Eq.(56),

$$\sum_{\dots\alpha_{\tau(k)}\alpha_{\tau(k+1)}\dots}^* = \sum_{\dots\alpha_{j+1}<\alpha_j\dots},$$

changes when we consider the summation in Eq.(69),

$$\sum_{\dots\alpha_{\tau(k)}\alpha_{\tau(k+1)}\dots}^{**} = \sum_{\dots\alpha_{j+1}\leq\alpha_j\dots}.$$

3.3.1 The case $|k - l| > 1$

Applying the above analysis we now consider the left-hand side of Eq.(67),

$$\sum_{\sigma \in S_L} \sum_{1 \leq \alpha_{\sigma(1)} \dots \alpha_{\sigma(L)} \leq N}^{**} \bigotimes_{i=1}^L e_{\sigma(i)}^{(\alpha_{\sigma(i)} \alpha_{\sigma(i)})} R_{1 \dots L}^{\{\sigma\}} - \mathcal{R}_{j(j+1)} \sum_{\sigma \in S_L} \sum_{1 \leq \alpha_{\sigma(1)} \dots \alpha_{\sigma(L)} \leq N}^* \bigotimes_{i=1}^L e_{\sigma(i)}^{(\alpha_{\sigma(i)} \alpha_{\sigma(i)})} R_{1 \dots L}^{\{\sigma\}} \quad (72)$$

and in particular, focus on the case where $\sigma(k) = j$, $\sigma(l) = j + 1$ and $|k - l| > 1$. In this case,

$\sum_{1 \leq \alpha_{\sigma(1)} \dots \alpha_{\sigma(L)} \leq N}^{**} = \sum_{1 \leq \alpha_{\sigma(1)} \dots \alpha_{\sigma(L)} \leq N}^*$, and hence Eq.(72) becomes,

$$(\mathcal{I}_j \otimes \mathcal{I}_{j+1} - \mathcal{R}_{j(j+1)}) \sum_{\substack{k, l=1 \\ |k-l|>1}}^L \sum_{\substack{\sigma \in S_L \\ \sigma(k)=j, \sigma(l)=j+1}} \sum_{1 \leq \alpha_{\sigma(1)} \dots \alpha_{\sigma(L)} \leq N}^* \bigotimes_{i=1}^L e_{\sigma(i)}^{(\alpha_{\sigma(i)} \alpha_{\sigma(i)})} R_{1 \dots L}^{\{\sigma\}}. \quad (73)$$

Through inspection one can see that the condition $|k - l| > 1$ means that $\alpha_j \neq \alpha_{j+1}$ in the summation - the easiest way to convince oneself of this statement is to consider the case $|k - l| = 2$ (for clarity label $l = k + 2$) and look at the inequality conditions between $\alpha_{\sigma(k)}$, $\alpha_{\sigma(k+1)}$ and $\alpha_{\sigma(k+2)}$. Hence, through applying Eq.(51), Eq.(73) becomes,

$$\sum_{\substack{k, l=1 \\ |k-l|>1}}^L \sum_{\substack{\sigma \in S_L \\ \sigma(k)=j, \sigma(l)=j+1}} \sum_{1 \leq \alpha_{\sigma(1)} \dots \alpha_{\sigma(L)} \leq N}^* \overbrace{(\mathcal{I}_j \otimes \mathcal{I}_{j+1} - \mathcal{R}_{j(j+1)}) \left\{ e_j^{(\alpha_j \alpha_j)} \otimes e_{j+1}^{(\alpha_{j+1} \alpha_{j+1})} \right\}}^{\text{apply Eq.(51)}} \bigotimes_{\substack{i=1 \\ i \neq k, l}}^L e_{\sigma(i)}^{(\alpha_{\sigma(i)} \alpha_{\sigma(i)})} R_{1 \dots L}^{\{\sigma\}} = 0.$$

3.3.2 The case $|k - l| = 1$

We now focus on the case $\sigma(k) = j$, $\sigma(l) = j + 1$ where $|k - l| = 1$. In this case $\sum_{1 \leq \alpha_{\sigma(1)} \dots \alpha_{\sigma(L)} \leq N}^{**} \neq \sum_{1 \leq \alpha_{\sigma(1)} \dots \alpha_{\sigma(L)} \leq N}^*$, hence we look at each term in Eq.(72) separately,

$$\begin{aligned} & \sum_{\substack{k, l=1 \\ |k-l|=1}}^L \sum_{\substack{\sigma \in S_L \\ \sigma(k)=j, \sigma(l)=j+1}} \sum_{1 \leq \alpha_{\sigma(1)} \dots \alpha_{\sigma(L)} \leq N}^{**} \bigotimes_{i=1}^L e_{\sigma(i)}^{(\alpha_{\sigma(i)} \alpha_{\sigma(i)})} R_{1 \dots L}^{\{\sigma\}} \\ &= \sum_{k=1}^L \sum_{\substack{\sigma \in S_L \\ \sigma(k)=j, \sigma(k+1)=j+1}} \sum_{1 \leq \alpha_{\sigma(1)} \dots \alpha_j < \alpha_{j+1} \dots \alpha_{\sigma(L)} \leq N}^* \bigotimes_{i=1}^L e_{\sigma(i)}^{(\alpha_{\sigma(i)} \alpha_{\sigma(i)})} R_{1 \dots L}^{\{\sigma\}} \end{aligned} \quad (74)$$

$$+ \sum_{k=1}^L \sum_{\substack{\sigma \in S_L \\ \sigma(k+1)=j, \sigma(k)=j+1}} \sum_{1 \leq \alpha_{\sigma(1)} \dots \alpha_{j+1} \leq \alpha_j \dots \alpha_{\sigma(L)} \leq N}^* \bigotimes_{i=1}^L e_{\sigma(i)}^{(\alpha_{\sigma(i)} \alpha_{\sigma(i)})} R_{1 \dots L}^{\{\sigma\}}, \quad (75)$$

and,

$$\begin{aligned} & \mathcal{R}_{j(j+1)} \sum_{\substack{k,l=1 \\ |k-l|=1}}^L \sum_{\substack{\sigma \in S_L \\ \sigma(k)=j, \sigma(l)=j+1}} \sum_{1 \leq \alpha_{\sigma(1)} \dots \alpha_{\sigma(L)} \leq N}^* \bigotimes_{i=1}^L e_{\sigma(i)}^{(\alpha_{\sigma(i)} \alpha_{\sigma(i)})} R_{1\dots L}^{\{\sigma\}} \\ = & \mathcal{R}_{j(j+1)} \sum_{k=1}^L \sum_{\substack{\sigma \in S_L \\ \sigma(k)=j, \sigma(k+1)=j+1}} \sum_{1 \leq \alpha_{\sigma(1)} \dots \alpha_j \leq \alpha_{j+1} \dots \alpha_{\sigma(L)} \leq N}^* \bigotimes_{i=1}^L e_{\sigma(i)}^{(\alpha_{\sigma(i)} \alpha_{\sigma(i)})} R_{1\dots L}^{\{\sigma\}} \end{aligned} \quad (76)$$

$$+ \mathcal{R}_{j(j+1)} \sum_{k=1}^L \sum_{\substack{\sigma \in S_L \\ \sigma(k+1)=j, \sigma(k)=j+1}} \sum_{1 \leq \alpha_{\sigma(1)} \dots \alpha_{j+1} < \alpha_j \dots \alpha_{\sigma(L)} \leq N}^* \bigotimes_{i=1}^L e_{\sigma(i)}^{(\alpha_{\sigma(i)} \alpha_{\sigma(i)})} R_{1\dots L}^{\{\sigma\}}. \quad (77)$$

We notice that the summations over the α_i 's in Eqs. (74) and (75) are now of the same type as those in Eqs. (76) and (77) - i.e. they have only one “*” symbol. This is because the summation indices which cause the difference (α_j and α_{j+1}) have been dealt with explicitly.

We now subtract the $\alpha_j \neq \alpha_{j+1}$ component of Eq.(76) from Eq.(74) to obtain,

$$\begin{aligned} & \sum_{k=1}^L \sum_{\substack{\sigma \in S_L \\ \sigma(k)=j, \sigma(k+1)=j+1}} \sum_{1 \leq \alpha_{\sigma(1)} \dots \alpha_j < \alpha_{j+1} \dots \alpha_{\sigma(L)} \leq N}^* \overbrace{(\mathcal{I}_j \otimes \mathcal{I}_{j+1} - \mathcal{R}_{j(j+1)})}^{\text{apply Eq.(51)}} \left\{ e_j^{(\alpha_j \alpha_j)} \otimes e_{j+1}^{(\alpha_{j+1} \alpha_{j+1})} \right\} \\ & \bigotimes_{\substack{i=1 \\ i \neq k, k+1}}^L e_{\sigma(i)}^{(\alpha_{\sigma(i)} \alpha_{\sigma(i)})} R_{1\dots L}^{\{\sigma\}} = 0. \end{aligned} \quad (78)$$

Similarly we subtract Eq.(77) from the $\alpha_j \neq \alpha_{j+1}$ component of Eq.(75) to obtain,

$$\begin{aligned} & \sum_{k=1}^L \sum_{\substack{\sigma \in S_L \\ \sigma(k+1)=j, \sigma(k)=j+1}} \sum_{1 \leq \alpha_{\sigma(1)} \dots \alpha_{j+1} < \alpha_j \dots \alpha_{\sigma(L)} \leq N}^* \overbrace{(\mathcal{I}_j \otimes \mathcal{I}_{j+1} - \mathcal{R}_{j(j+1)})}^{\text{apply Eq.(51)}} \left\{ e_j^{(\alpha_j \alpha_j)} \otimes e_{j+1}^{(\alpha_{j+1} \alpha_{j+1})} \right\} \\ & \bigotimes_{\substack{i=1 \\ i \neq k, k+1}}^L e_{\sigma(i)}^{(\alpha_{\sigma(i)} \alpha_{\sigma(i)})} R_{1\dots L}^{\{\sigma\}} = 0. \end{aligned} \quad (79)$$

Finally, we now subtract the $\alpha_j = \alpha_{j+1}$ component of Eq.(76) from the $\alpha_j = \alpha_{j+1}$ component of Eq.(75) to obtain,

$$\begin{aligned} & \sum_{k=1}^L \sum_{\substack{\tau \in S_L \\ \tau(k+1)=j, \tau(k)=j+1}} \sum_{1 \leq \alpha_{\tau(1)} \dots \alpha_{j+1} = \alpha_j \dots \alpha_{\tau(L)} \leq N}^* \bigotimes_{i=1}^L e_{\tau(i)}^{(\alpha_{\tau(i)} \alpha_{\tau(i)})} R_{1\dots L}^{\{\tau\}} \\ & - \mathcal{R}_{j(j+1)} \sum_{k=1}^L \sum_{\substack{\sigma \in S_L \\ \sigma(k)=j, \sigma(k+1)=j+1}} \sum_{1 \leq \alpha_{\sigma(1)} \dots \alpha_j = \alpha_{j+1} \dots \alpha_{\sigma(L)} \leq N}^* \bigotimes_{i=1}^L e_{\sigma(i)}^{(\alpha_{\sigma(i)} \alpha_{\sigma(i)})} R_{1\dots L}^{\{\sigma\}}. \end{aligned} \quad (80)$$

We proceed in Eq.(80) by first making the change $\tau = \sigma\sigma_{k(k+1)}$ in permutation labels. Next we apply Eq.(41) to the permuted R -matrix $R_{1\dots L}^{\{\tau\}}$ to obtain,

$$R_{1\dots L}^{\{\tau\}} = R_{\sigma(1\dots L)}^{\{\sigma_{k(k+1)}\}} R_{1\dots L}^{\{\sigma\}} = R_{j(j+1)} R_{1\dots L}^{\{\sigma\}}. \quad (81)$$

Taking into account the above change in permutation labels and decomposition of the permuted R -matrix, Eq.(80) becomes,

$$\sum_{k=1}^L \sum_{\substack{\sigma \in S_L \\ \sigma(k)=j, \sigma(k+1)=j+1}} \sum_{1 \leq \alpha_{\sigma(1)} \dots \alpha_j = \alpha_{j+1} \dots \alpha_{\sigma(L)} \leq N} \bigotimes_{\substack{i=1 \\ i \neq k, k+1}}^L e_{\sigma(i)}^{(\alpha_{\sigma(i)} \alpha_{\sigma(i)})} \underbrace{\left\{ \left(e_j^{(\alpha_j \alpha_j)} \otimes e_{j+1}^{(\alpha_{j+1} \alpha_{j+1})} \right) R_{j(j+1)} - \mathcal{R}_{j(j+1)} \left(e_j^{(\alpha_j \alpha_j)} \otimes e_{j+1}^{(\alpha_{j+1} \alpha_{j+1})} \right) \right\}}_{\text{apply Eq.(51)}} R_{1\dots L}^{\{\sigma\}} = 0. \quad (82)$$

Hence we have verified the factorization condition for general L and N .

4 Twisted monodromy operators for $N = 3$

The purpose of this section is to show that the constructed F -matrices can effectively be used as similarity transformation in the simplest situation of the $N = 3$ state vertex model. We shall present an algebraic derivation of the form of some relevant elements of the monodromy matrix in the F -basis for arbitrary weights. Here we represent the $N = 3$ monodromy matrix as,

$$\mathcal{T}_{a,1\dots L}(\mu) = \begin{pmatrix} A_{1\dots L}^{(11)}(\mu) & A_{1\dots L}^{(12)}(\mu) & B_{1\dots L}^{(1)}(\mu) \\ A_{1\dots L}^{(21)}(\mu) & A_{1\dots L}^{(22)}(\mu) & B_{1\dots L}^{(2)}(\mu) \\ C_{1\dots L}^{(1)}(\mu) & C_{1\dots L}^{(2)}(\mu) & D_{1\dots L}(\mu) \end{pmatrix}_a. \quad (83)$$

In general, a given monodromy matrix element $X_{1\dots L}(\mu)$ can be transformed to a new operator $\tilde{X}_{1\dots L}(\mu)$ having a much simpler quasilocal form with the help of the F -matrices [4]. Since $F_{1\dots L}$ and $\mathcal{F}_{1\dots L}$ are related via the multiplication of a diagonal matrix (53), it is enough to compute the following non-trivial twisted operators,

$$\tilde{X}_{1\dots L}(\mu) = \mathcal{F}_{1\dots L} X_{1\dots L}(\mu) \mathcal{F}_{1\dots L}^{-1}. \quad (84)$$

In what follows we shall present a conjecture for the expressions of the twisted monodromy operators $\tilde{D}_{1\dots L}(\mu)$, $\tilde{C}_{1\dots L}^{(2)}(\mu)$, $\tilde{B}_{1\dots L}^{(2)}(\mu)$, $\tilde{C}_{1\dots L}^{(1)}(\mu)$ and $\tilde{B}_{1\dots L}^{(1)}(\mu)$. Our results are built up from an analysis performed in the cases $L = 2$ and $L = 3$ relying only on the identities derived from the Yang-Baxter and unitarity relations. Fortunately, this study is sufficient to foresee the main structure of the mentioned twisted operators for arbitrary L and general Boltzmann weights without relying on any specific parameterizations.

We now list the final expressions for the above mentioned twisted operators,

$$\tilde{D}_{1\dots L}(\mu) = \bigotimes_{i=1}^L \text{diag} \{b_{31}(\mu, \xi_i), b_{32}(\mu, \xi_i), a_3(\mu, \xi_i)\}_i \quad (85)$$

$$\tilde{C}_{1\dots L}^{(2)}(\mu) = \sum_{l=1}^L c_{32}(\mu, \xi_l) e_l^{(23)} \bigotimes_{\substack{i=1 \\ i \neq l}}^L \text{diag} \left\{ b_{21}(\mu, \xi_i), \frac{b_{32}(\mu, \xi_i)}{b_{32}(\xi_l, \xi_i) \theta_2(\xi_i, \xi_l)}, a_3(\mu, \xi_i) \theta_3(\xi_i, \xi_l) \right\}_i \quad (86)$$

$$\tilde{B}_{1\dots L}^{(2)}(\mu) = \sum_{l=1}^L c_{23}(\mu, \xi_l) e_l^{(32)} \bigotimes_{\substack{i=1 \\ i \neq l}}^L \text{diag} \left\{ b_{31}(\mu, \xi_i), b_{32}(\mu, \xi_i) \theta_2(\xi_l, \xi_i), \frac{a_3(\mu, \xi_i)}{b_{32}(\xi_i, \xi_l) \theta_3(\xi_l, \xi_i)} \right\}_i \quad (87)$$

$$\begin{aligned} \tilde{C}_{1\dots L}^{(1)}(\mu) &= \sum_{l=1}^L c_{31}(\mu, \xi_l) e_l^{(13)} \bigotimes_{\substack{i=1 \\ i \neq l}}^L \text{diag} \left\{ \frac{b_{21}(\mu, \xi_i)}{b_{21}(\xi_l, \xi_i) \theta_1(\xi_i, \xi_l)}, \frac{b_{32}(\mu, \xi_i)}{b_{32}(\xi_l, \xi_i)}, a_3(\mu, \xi_i) \theta_3(\xi_i, \xi_l) \right\}_i \\ &+ \sum_{\substack{l_1, l_2=1 \\ l_1 \neq l_2}}^L \frac{c_{31}(\mu, \xi_{l_1}) b_{32}(\mu, \xi_{l_2}) c_{21}(\xi_{l_1}, \xi_{l_2})}{b_{32}(\xi_{l_1}, \xi_{l_2})} e_{l_1}^{(23)} \otimes e_{l_2}^{(12)} \\ &\bigotimes_{\substack{i=1 \\ i \neq l_1, l_2}}^L \text{diag} \left\{ \frac{b_{21}(\mu, \xi_i)}{b_{21}(\xi_{l_2}, \xi_i) \theta_1(\xi_i, \xi_{l_2})}, \frac{b_{32}(\mu, \xi_i) \theta_2(\xi_i, \xi_{l_2})}{b_{32}(\xi_{l_1}, \xi_i) \theta_2(\xi_i, \xi_{l_1})}, a_3(\mu, \xi_i) \theta_3(\xi_i, \xi_{l_1}) \right\}_i \end{aligned} \quad (88)$$

$$\begin{aligned} \tilde{B}_{1\dots L}^{(1)}(\mu) &= \sum_{l=1}^L c_{13}(\mu, \xi_l) e_l^{(31)} \bigotimes_{\substack{i=1 \\ i \neq l}}^L \text{diag} \left\{ b_{31}(\mu, \xi_i) \theta_1(\xi_l, \xi_i), \frac{b_{32}(\mu, \xi_i)}{b_{21}(\xi_i, \xi_l)}, \frac{a_3(\mu, \xi_i)}{b_{31}(\xi_i, \xi_l) \theta_3(\xi_l, \xi_i)} \right\}_i \\ &+ \sum_{\substack{l_1, l_2=1 \\ l_1 \neq l_2}}^L \frac{c_{13}(\mu, \xi_{l_1}) b_{32}(\mu, \xi_{l_2}) c_{12}(\xi_{l_1}, \xi_{l_2})}{b_{21}(\xi_{l_1}, \xi_{l_2})} e_{l_1}^{(32)} \otimes e_{l_2}^{(21)} \\ &\bigotimes_{\substack{i=1 \\ i \neq l_1, l_2}}^L \text{diag} \left\{ b_{31}(\mu, \xi_i) \theta_1(\xi_{l_2}, \xi_i), \frac{b_{32}(\mu, \xi_i) \theta_2(\xi_{l_1}, \xi_i)}{b_{21}(\xi_i, \xi_{l_2}) \theta_2(\xi_{l_2}, \xi_i)}, \frac{a_3(\mu, \xi_i)}{b_{31}(\xi_i, \xi_{l_1}) \theta_3(\xi_{l_1}, \xi_i)} \right\}_i \end{aligned} \quad (89)$$

where the auxiliary functions $\theta_i(\xi_j, \xi_k)$ are given by,

$$\theta_i(\xi_j, \xi_k) = \begin{cases} a_i(\xi_j, \xi_k) & \text{for } j < k \\ 1 & \text{for } j \geq k \end{cases} \quad i = 1, 2, 3. \quad (90)$$

We begin by explicitly verifying that the $L = 2$ case holds for each of the twisted operators given below.

4.1 The $L=2$ case

In this section we shall calculate the twisted operators $\tilde{D}_{12}(\mu)$, $\tilde{C}_{12}^{(2)}(\mu)$, $\tilde{B}_{12}^{(2)}(\mu)$, $\tilde{C}_{12}^{(1)}(\mu)$ and $\tilde{B}_{12}^{(1)}(\mu)$ directly from the similarity transform (84). We shall then detail the necessary Yang-Baxter and unitary relations for the entries obtained from Eq.(84) to match the corresponding entries in the $L = 2$ operators given by Eqs.(85-89). The expressions derived directly from Eq.(84) have the following structure,

$$\begin{aligned} \tilde{D}_{12}(\mu) &= \text{diag}\{b_{31}(\mu, \xi_1), b_{32}(\mu, \xi_1), a_3(\mu, \xi_1)\}_1 \otimes \text{diag}\{b_{31}(\mu, \xi_2), b_{32}(\mu, \xi_2), a_3(\mu, \xi_2)\}_2 \\ &\quad + \kappa_1^{(D)} e_1^{(21)} \otimes e_2^{(12)} + \kappa_2^{(D)} e_1^{(31)} \otimes e_2^{(13)} + \kappa_3^{(D)} e_1^{(32)} \otimes e_2^{(23)} \\ \tilde{C}_{12}^{(2)}(\mu) &= \text{diag}\{\kappa_1^{(C_2)}, \kappa_2^{(C_2)}, \kappa_3^{(C_2)}\}_1 \otimes e_2^{(23)} + e_1^{(23)} \otimes \text{diag}\{\kappa_4^{(C_2)}, \kappa_5^{(C_2)}, \kappa_6^{(C_2)}\}_2 \\ &\quad + \kappa_7^{(C_2)} e_1^{(21)} \otimes e_2^{(13)} \\ \tilde{B}_{12}^{(2)}(\mu) &= \text{diag}\{\kappa_1^{(B_2)}, \kappa_2^{(B_2)}, \kappa_3^{(B_2)}\}_1 \otimes e_2^{(32)} + e_1^{(32)} \otimes \text{diag}\{\kappa_4^{(B_2)}, \kappa_5^{(B_2)}, \kappa_6^{(B_2)}\}_2 \\ &\quad + \kappa_7^{(B_2)} e_1^{(31)} \otimes e_2^{(12)} \end{aligned} \quad (91)$$

$$\begin{aligned} \tilde{C}_{12}^{(1)}(\mu) &= \text{diag}\{\kappa_1^{(C_1)}, \kappa_2^{(C_1)}, \kappa_3^{(C_1)}\}_1 \otimes e_2^{(13)} + e_1^{(13)} \otimes \text{diag}\{\kappa_4^{(C_1)}, \kappa_5^{(C_1)}, \kappa_6^{(C_1)}\}_2 \\ &\quad + \kappa_7^{(C_1)} e_1^{(12)} \otimes e_2^{(23)} + \kappa_8^{(C_1)} e_1^{(23)} \otimes e_2^{(12)} \\ \tilde{B}_{12}^{(1)}(\mu) &= \text{diag}\{\kappa_1^{(B_1)}, \kappa_2^{(B_1)}, \kappa_3^{(B_1)}\}_1 \otimes e_2^{(31)} + e_1^{(31)} \otimes \text{diag}\{\kappa_4^{(B_1)}, \kappa_5^{(B_1)}, \kappa_6^{(B_1)}\}_2 \\ &\quad + \kappa_7^{(B_1)} e_1^{(21)} \otimes e_2^{(32)} + \kappa_8^{(B_1)} e_1^{(32)} \otimes e_2^{(21)} \end{aligned} \quad (92)$$

All the κ entries of the above matrices do not immediately match the corresponding entries calculated from the $L = 2$ expressions of Eqs.(85)-(89). It is possible to simplify these entries using only certain relations coming from the Yang-Baxter (1) and unitarity (2) relations. The technical details are quite cumbersome and thus have been deferred to Appendix B. In Table (1)

Operator	Y-B equations	Unitarity equations
\tilde{D}_{12}	(12)-{3,1,2} (14)-{3,1}-{3,2}	
$\tilde{C}_{12}^{(2)}$	(13)-{2,3,1} (16)-{3,2} (19)-{3,2} (21)-{3,2,1}	(10)-{3,2}
$\tilde{B}_{12}^{(2)}$	(13)-{3,2,1} (17)-{2,3} (18)-{2,3} (21)-{2,3,1}	(10)-{2,3}
$\tilde{C}_{12}^{(1)}$	(13)-{1,3,2} (16)-{3,1} (19)-{3,1} (20)-{3,2,1}	(9)-{2,1} (10)-{3,2}
$\tilde{B}_{12}^{(1)}$	(13)-{1,3,2} (17)-{1,3} (18)-{1,3} (21)-{1,3,2}	(9)-{1,2} (10)-{2,1}-{3,1}

Table 1: Required Yang-Baxter and unitarity relations for $L = 2$.

we provide a summary of the unitarity and Yang-Baxter relations that are required to simplify such non-trivial entries in order to bring the operators in the form given by Eqs.(85)-(89).

We remark that in this table a given algebraic relation among weights is referred to the equation number together with its respective indices $\{i\}$, $\{i, j\}$ or $\{i, j, k\}$. For example, the symbol (9)-{1, 3} refers to the equation $b_{13}(\xi_1, \xi_2)b_{31}(\xi_2, \xi_1) + c_{13}(\xi_1, \xi_2)c_{31}(\xi_2, \xi_1) = 1$ whilst (12)-{1, 3, 2} means the relation $b_{13}(\xi_1, \xi_2)b_{12}(\xi_1, \xi_3) = b_{12}(\xi_1, \xi_2)b_{13}(\xi_1, \xi_3)$. We observe that there exists obvious equivalences for some equations such as for Eq.(12) the indices $\{i, j, k\}$ or $\{i, k, j\}$ leads to the same relation.

At this point we remark that the $L = 2$ case does not allow one to verify the entries of the diagonal matrices of the double summation from $\tilde{C}_{1\dots L}^{(1)}$ and $\tilde{B}_{1\dots L}^{(1)}$, we have only verified the coefficients of the double summation terms, given by $\frac{c_{31}(\mu, \xi_{l_1})b_{32}(\mu, \xi_{l_2})c_{21}(\xi_{l_1}, \xi_{l_2})}{b_{32}(\xi_{l_1}, \xi_{l_2})}$ and $\frac{c_{13}(\mu, \xi_{l_1})b_{32}(\mu, \xi_{l_2})c_{12}(\xi_{l_1}, \xi_{l_2})}{b_{21}(\xi_{l_1}, \xi_{l_2})}$ respectively. In order to obtain the entries of the aforementioned diagonal matrices we need to consider the $L = 3$ case.

4.2 The case L=3

We now consider explicitly calculating the $L = 3$ case using Eq.(84). Here it is only necessary to consider the operators $\tilde{C}_{123}^{(1)}(\mu)$ and $\tilde{B}_{123}^{(1)}(\mu)$ and compute the entries that are completely missed

by the $L = 2$ case. We begin with the operator $\tilde{C}_{123}^{(1)}(\mu)$ whose entries are proportional to,

$$\begin{aligned} & e_1^{(12)} \otimes e_2^{(23)} \otimes \text{diag}\{\alpha_1^{(C_1)}, \alpha_2^{(C_1)}, \alpha_3^{(C_1)}\}_3 + e_1^{(12)} \otimes \text{diag}\{\beta_1^{(C_1)}, \beta_2^{(C_1)}, \beta_3^{(C_1)}\}_2 \otimes e_3^{(23)} \\ & + \text{diag}\{\gamma_1^{(C_1)}, \gamma_2^{(C_1)}, \gamma_3^{(C_1)}\}_1 \otimes e_2^{(12)} \otimes e_3^{(23)} + e_1^{(23)} \otimes e_2^{(12)} \otimes \text{diag}\{\delta_1^{(C_1)}, \delta_2^{(C_1)}, \delta_3^{(C_1)}\}_3 \\ & + e_1^{(23)} \otimes \text{diag}\{\phi_1^{(C_1)}, \phi_2^{(C_1)}, \phi_3^{(C_1)}\}_2 \otimes e_3^{(12)} + \text{diag}\{\omega_1^{(C_1)}, \omega_2^{(C_1)}, \omega_3^{(C_1)}\}_1 \otimes e_2^{(23)} \otimes e_3^{(12)}. \end{aligned} \quad (93)$$

As with the $L = 2$ case, we shall detail the Yang-Baxter and unitary relations necessary for the entries obtained from Eq.(84) to match the $L = 3$ case of Eq.(88). In Table (2) we provide a summary of the unitarity and Yang-Baxter relations that are required to simplify all the entries in order to bring the operators in the form given by Eq.(88). The technicalities of this calculation are rather involved and thus have been deferred to Appendix C.

We now turn our attention to the operator $\tilde{B}_{123}^{(1)}(\mu)$ whose entries are, completely missed by the $L = 2$ case are given by,

$$\begin{aligned} & e_1^{(32)} \otimes e_2^{(21)} \otimes \text{diag}\{\alpha_1^{(B_1)}, \alpha_2^{(B_1)}, \alpha_3^{(B_1)}\}_3 + e_1^{(32)} \otimes \text{diag}\{\beta_1^{(B_1)}, \beta_2^{(B_1)}, \beta_3^{(B_1)}\}_2 \otimes e_3^{(21)} \\ & + \text{diag}\{\gamma_1^{(B_1)}, \gamma_2^{(B_1)}, \gamma_3^{(B_1)}\}_1 \otimes e_2^{(32)} \otimes e_3^{(21)} + e_1^{(21)} \otimes e_2^{(32)} \otimes \text{diag}\{\delta_1^{(B_1)}, \delta_2^{(B_1)}, \delta_3^{(B_1)}\}_3 \\ & + e_1^{(21)} \otimes \text{diag}\{\phi_1^{(B_1)}, \phi_2^{(B_1)}, \phi_3^{(B_1)}\}_2 \otimes e_3^{(32)} + \text{diag}\{\omega_1^{(B_1)}, \omega_2^{(B_1)}, \omega_3^{(B_1)}\}_1 \otimes e_2^{(21)} \otimes e_3^{(32)}. \end{aligned}$$

The technical details entering in the simplifications of these entries are similar to that conducted for the operator $\tilde{C}_{123}^{(1)}(\mu)$. We therefore restrict ourselves in presenting only the required Yang-Baxter and unitarity relations that are necessary for the raw entries obtained from the $L = 3$ case of Eq.(84) to become the corresponding $L = 3$ entries in Eq.(89). This is summarized in Table (3).

4.3 The general L case

At present we just have a proof for arbitrary L in the case of simplest the twisted operator $\tilde{D}_{1\dots L}(\mu)$. We now present the proof of Eq.(85) for general L . Our verification is an adaptation of an argument first given in [15] for the equivalent operator. To begin we note that the operator $D_{1\dots L}(\mu)$ can be expressed through,

$$e_a^{(33)} D_{1\dots L}(\mu) = e_a^{(33)} \mathcal{T}_{a,1\dots L}(\mu) e_a^{(33)}, \quad (94)$$

Entry	Y-B equations	Unitarity equations
$\alpha_1^{(C_1)}$	(13)-{3,2,1} (20)-{3,2,1}	(10)-{3,2}
$\alpha_2^{(C_1)}$	(20)-{3,2,1}	(10)-{3,2}
$\alpha_3^{(C_1)}$	(20)-{3,2,1}	(10)-{3,2}
$\beta_1^{(C_1)}$	(13)-{3,2,1} (20)-{3,2,1}	(10)-{3,2}
$\beta_2^{(C_1)}$	(19)-{3,2} (20)-{3,2,1}	(10)-{3,2}
$\beta_3^{(C_1)}$	(14)-{3,2} (15)-{3,2} (16)-{3,2} (20)-{3,2,1}	(10)-{3,2}
$\gamma_1^{(C_1)}$	(13)-{3,2,1} (19)-{2,1}-{3,1} (20)-{3,2,1}	(10)-{2,1}-{3,1}-{3,2}
$\gamma_2^{(C_1)}$	(13)-{1,3,2} (16)-{2,1} (19)-{3,2} (20)-{3,2,1}	(10)-{3,2}
$\gamma_3^{(C_1)}$	(12)-{3,2,1} (15)-{3,2} (16)-{3,1} (20)-{3,2,1}	(10)-{3,2}
$\delta_1^{(C_1)}$	(13)-{3,2,1}	
$\delta_2^{(C_1)}$		
$\delta_3^{(C_1)}$		
$\phi_1^{(C_1)}$	(12)-{3,2,1} (13)-{3,2,1} (19)-{2,1}	(10)-{2,1}
$\phi_2^{(C_1)}$		
$\phi_3^{(C_1)}$		
$\omega_1^{(C_1)}$	(12)-{3,2,1} (13)-{2,3,1} (19)-{2,1}-{3,1} (20)-{3,2,1}	(10)-{2,1}-{3,1}
$\omega_2^{(C_1)}$	(13)-{1,3,2} (16)-{2,1} (20)-{3,2,1}	(10)-{3,2}
$\omega_3^{(C_1)}$	(12)-{3,2,1} (16)-{3,1} (20)-{3,2,1}	

Table 2: Required Yang-Baxter and unitarity relations for the entries of $\tilde{C}_{123}^{(1)}$.

Entry	Y-B equations	Unitarity equations
$\alpha_1^{(B_1)}$	(13)-{2,3,1} (18)-{1,2}-{1,3} (21)-{1,3,2}	(8)-{1}
$\alpha_2^{(B_1)}$	(13)-{1,3,2} (17)-{1,2} (18)-{2,3} (21)-{1,3,2}	(8)-{2} (10)-{2,1}
$\alpha_3^{(B_1)}$	(12) - {3, 1, 2}(13) - {2, 3, 1} (14) - {3, 2}(17) - {1, 3}(21) - {1, 3, 2}	(10)-{2,1}-{3,1}-{3,2}
$\beta_1^{(B_1)}$	(13)-{2,3,1} (21)-{1,3,2}	(10)-{3,2}
$\beta_2^{(B_1)}$	(18)-{2,3} (21)-{1,3,2}	
$\beta_3^{(B_1)}$	(12) - {3, 1, 2}(13) - {2, 3, 1}(14) - {3, 2} (15) - {3, 2}(17) - {2, 3}(21) - {1, 3, 2}	(10)-{2,1}
$\gamma_1^{(B_1)}$	(21)-{1,3,2}	
$\gamma_2^{(B_1)}$	(21)-{1,3,2}	
$\gamma_3^{(B_1)}$	(12)-{3,1,2} (21)-{1,3,2}	
$\delta_1^{(B_1)}$	(12)-{3,1,2} (18)-{1,2}-{1,3} (21)-{1,3,2}	(8)-{1} (10)-{2,1}
$\delta_2^{(B_1)}$	(13)-{1,3,2} (14)-{2,1} (17)-{1,2} (21)-{1,3,2}	(8)-{2} (10)-{2,1}
$\delta_3^{(B_1)}$	(17)-{1,3}	(10)-{2,1}-{3,1}
$\phi_1^{(B_1)}$	(12)-{3,1,2}	(10)-{2,1}
$\phi_2^{(B_1)}$	(14)-{2,1}	(10)-{2,1}
$\phi_3^{(B_1)}$		(10)-{2,1}
$\omega_1^{(B_1)}$		(10)-{2,1}
$\omega_2^{(B_1)}$		(10)-{2,1}
$\omega_3^{(B_1)}$	(12)-{3,1,2}	(10)-{2,1}

Table 3: Required Yang-Baxter and unitarity relations for the entries of $\tilde{B}_{123}^{(1)}$.

where the Weyl matrices $e_a^{(33)}$ project out the operator $D_{1\dots L}(\mu)$ from the 3×3 matrix expression in auxiliary space \mathcal{A}_a .

Using the above expression we now consider the action of the \mathcal{F} -matrix on $D_{1\dots L}(\mu)$,

$$\begin{aligned} e_a^{(33)} \mathcal{F}_{1\dots L} D_{1\dots L}(\mu) &= \sum_{\sigma \in S_L} \sum_{1 \leq \alpha_{\sigma(1)} \dots \alpha_{\sigma(L)} \leq 3}^* \bigotimes_{i=1}^L e_{\sigma(i)}^{(\alpha_{\sigma(i)} \alpha_{\sigma(i)})} e_a^{(33)} \overbrace{R_{1\dots L}^{\{\sigma\}} \mathcal{T}_{a,1\dots L}(\mu)}^{\text{apply Eq.(43)}} e_a^{(33)} \\ &= \sum_{\sigma \in S_L} \sum_{1 \leq \alpha_{\sigma(1)} \dots \alpha_{\sigma(L)} \leq 3}^* \bigotimes_{i=1}^L e_{\sigma(i)}^{(\alpha_{\sigma(i)} \alpha_{\sigma(i)})} e_a^{(33)} \mathcal{T}_{a,\sigma(1\dots L)}(\mu) e_a^{(33)} R_{1\dots L}^{\{\sigma\}}. \end{aligned} \quad (95)$$

In what follows, we separate the sum over the indices $\alpha_{\sigma(i)}$ according to the number of occurrences where $\alpha_{\sigma(i)} = 3$,

$$e_a^{(33)} \mathcal{F}_{1\dots L} D_{1\dots L}(\mu) = \sum_{\sigma \in S_L} \sum_{k=0}^L \sum_{1 \leq \alpha_{\sigma(1)} \dots \alpha_{\sigma(L)} \leq 3}^{*(k)} \bigotimes_{i=1}^L e_{\sigma(i)}^{(\alpha_{\sigma(i)} \alpha_{\sigma(i)})} e_a^{(33)} \mathcal{T}_{a,\sigma(1\dots L)}(\mu) e_a^{(33)} R_{1\dots L}^{\{\sigma\}}, \quad (96)$$

where,

$$\sum_{1 \leq \alpha_{\sigma(1)} \dots \alpha_{\sigma(L)} \leq 3}^{*(k)} = \sum_{\substack{1 \leq \alpha_{\sigma(1)} \dots \alpha_{\sigma(L)} \leq 3 \\ (\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(L-k)} \in \{1,2\}, (\alpha_{\sigma(L-k+1)}, \dots, \alpha_{\sigma(L)}) = 3}}^* . \quad (97)$$

Hence the tensor product of the Weyl matrices $\bigotimes_{i=1}^L e_{\sigma(i)}^{(\alpha_{\sigma(i)} \alpha_{\sigma(i)})}$ for each value of the index k becomes,

$$\bigotimes_{i=1}^L e_{\sigma(i)}^{(\alpha_{\sigma(i)} \alpha_{\sigma(i)})} = \bigotimes_{i=1}^{L-k} e_{\sigma(i)}^{(\alpha_{\sigma(i)} \alpha_{\sigma(i)})} \bigotimes_{i=L-k+1}^L e_{\sigma(i)}^{(33)},$$

leading to the following form for Eq.(96),

$$\begin{aligned} e_a^{(33)} \mathcal{F}_{1\dots L} D_{1\dots L}(\mu) &= \sum_{\sigma \in S_L} \sum_{k=0}^L \sum_{1 \leq \alpha_{\sigma(1)} \dots \alpha_{\sigma(L)} \leq 3}^{*(k)} \bigotimes_{i=1}^L e_{\sigma(i)}^{(\alpha_{\sigma(i)} \alpha_{\sigma(i)})} \\ &\quad \times e_a^{(33)} e_{\sigma(L)}^{(33)} R_{a\sigma(L)}(\mu) \dots e_{\sigma(L-k+1)}^{(33)} R_{a\sigma(L-k+1)}(\mu) \mathcal{T}_{a,\sigma(1\dots(L-k))}(\mu) e_a^{(33)} R_{1\dots L}^{\{\sigma\}}. \end{aligned} \quad (98)$$

In the above expression we realize that the action of the Weyl operators on the R -matrices leads to,

$$e_a^{(33)} e_{\sigma(L)}^{(33)} R_{a\sigma(L)}(\mu) \dots e_{\sigma(L-k+1)}^{(33)} R_{a\sigma(L-k+1)}(\mu) = \prod_{i=L-k+1}^L a_3(\mu, \xi_{\sigma(i)}) e_a^{(33)} \bigotimes_{j=L-k+1}^L e_{\sigma(j)}^{(33)},$$

and hence Eq.(98) becomes,

$$e_a^{(33)} \mathcal{F}_{1\dots L} D_{1\dots L}(\mu) = \sum_{\sigma \in S_L} \sum_{k=0}^L \sum_{1 \leq \alpha_{\sigma(1)} \dots \alpha_{\sigma(L)} \leq 3}^{*(k)} \prod_{i=L-k+1}^L a_3(\mu, \xi_{\sigma(i)}) e_a^{(33)} \\ \times e_{\sigma(L-k)}^{(\alpha_{\sigma(L-k)} \alpha_{\sigma(L-k)})} R_{a\sigma(L-k)}(\mu) \dots e_{\sigma(1)}^{(\alpha_{\sigma(1)} \alpha_{\sigma(1)})} R_{a\sigma(1)}(\mu) e_a^{(33)} \bigotimes_{j=L-k+1}^L e_{\sigma(j)}^{(33)} R_{1\dots L}^{\{\sigma\}}.$$

Since $\alpha_{\sigma(i)} \in \{1, 2\}$, $i = (1, \dots, L-k)$, in the above expression we realize that the action of the Weyl operators on the R -matrices leads to the following simplified expression,

$$e_a^{(33)} e_{\sigma(L-k)}^{(\alpha_{\sigma(L-k)} \alpha_{\sigma(L-k)})} R_{a\sigma(L-k)}(\mu) \dots e_{\sigma(1)}^{(\alpha_{\sigma(1)} \alpha_{\sigma(1)})} R_{a\sigma(1)}(\mu) e_a^{(33)} \\ = e_a^{(33)} \prod_{i=1}^{L-k} b_{3\alpha_{\sigma(i)}}(\mu, \xi_{\sigma(i)}) \bigotimes_{j=1}^{L-k} e_{\sigma(j)}^{(\alpha_{\sigma(j)} \alpha_{\sigma(j)})}.$$

The above considerations lead the following result,

$$\mathcal{F}_{1\dots L} D_{1\dots L}(\mu) = \sum_{\sigma \in S_L} \sum_{k=0}^L \sum_{1 \leq \alpha_{\sigma(1)} \dots \alpha_{\sigma(L)} \leq 3}^{*(k)} \prod_{i=1}^{L-k} b_{3\alpha_{\sigma(i)}}(\mu, \xi_{\sigma(i)}) \\ \prod_{j=L-k+1}^L a_3(\mu, \xi_{\sigma(j)}) \bigotimes_{l=1}^L e_l^{(\alpha_l \alpha_l)} R_{1\dots L}^{\{\sigma\}} \\ = \bigotimes_{i=1}^L \text{diag} \{b_{31}(\mu, \xi_i), b_{32}(\mu, \xi_i), a_3(\mu, \xi_i)\}_i \mathcal{F}_{1\dots L},$$

ultimately verifying Eq.(85) for general L .

We do not have a proof for general L in the cases of the twisted B and C operators. The necessary underlying recurrence relations to carry out such demonstrations have thus far eluded us. However, with the help of the Yang-Baxter solution presented in Section 2 we have been able to verify Eqs.(86-89) for $L = 4$ explicitly. Considering the generality of such solution, this is strong evidence supporting our conjectured expressions Eqs.(86-89) for general L .

5 Basic domain wall partition functions

The purpose of this section is to start the formulation to compute certain domain wall partition functions (DWPF's) associated to the $N = 3$ vertex model with arbitrary Boltzmann weights. The domain wall boundary conditions correspond to certain fixed statistical configurations for the horizontal and vertical edges at the top and bottom of the square lattice, see for instance [9, 26, 27].

From an algebraic perspective these objects can be expressed in terms of the expectation values of combinations of the creation and annihilation operators $B_{1\dots L}^{(i)}(\mu)$ and $C_{1\dots L}^{(i)}(\nu)$ on some pseudo-vacuum states [6, 9, 28, 29]. In Figures 2 and 3 we have depicted the graphical representation of such domain wall partition functions.

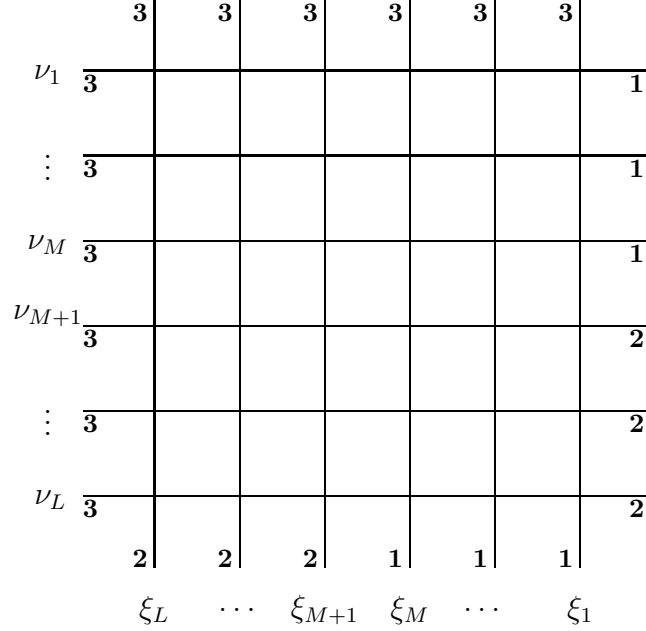


Figure 2: Graphical representation of a mixed DWPF of type C.

In order to compute such partition functions we need to first define the corresponding reference states as well as their properties under the action of the F -matrix. The $N = 3$ state vertex model has three possible pseudovacuum states which we label as follows,

$$|1\rangle_\alpha = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_\alpha, \quad |2\rangle_\alpha = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_\alpha, \quad |3\rangle_\alpha = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_\alpha, \quad (99)$$

for $\alpha = 1, \dots, L$ while the corresponding transpose states are denoted by,

$${}_\alpha\langle 1| = (1, 0, 0)_\alpha, \quad {}_\alpha\langle 2| = (0, 1, 0)_\alpha, \quad {}_\alpha\langle 3| = (0, 0, 1)_\alpha. \quad (100)$$

We also use the following convenient notational conventions with the outer products of reference

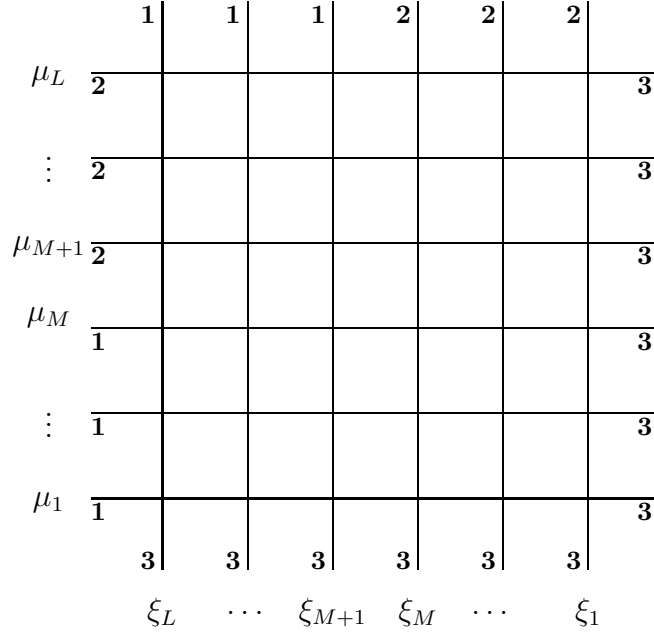


Figure 3: Graphical representation of a mixed DWPF of type B.

states,

$$|i\rangle_{\alpha_1 \dots \alpha_p} = \bigotimes_{j=1}^p |i\rangle_{\alpha_j} \quad i = (1, 2, 3),$$

and similarly with the transpose reference states.

One then finds that the action of a Weyl basis element on a reference state and a transpose reference state respectively is given by,

$$e_{\alpha}^{(ij)} |k\rangle_{\alpha} = |i\rangle_{\alpha} \delta_{jk} \quad , \quad {}_{\alpha} \langle i | e_{\alpha}^{(jk)} = {}_{\alpha} \langle k | \delta_{ij}. \quad (101)$$

Using the above observation we give the following results:

Proposition 2.

$$\begin{aligned} \mathcal{F}_{1\dots L} |i\rangle_{1\dots L} &= |i\rangle_{1\dots L} \quad , \quad {}_{1\dots L} \langle i | \mathcal{F}_{1\dots L} = {}_{1\dots L} \langle i | \\ \mathcal{F}_{1\dots L}^{-1} |i\rangle_{1\dots L} &= |i\rangle_{1\dots L} \quad , \quad {}_{1\dots L} \langle i | \mathcal{F}_{1\dots L}^{-1} = {}_{1\dots L} \langle i | \end{aligned} \quad (102)$$

Proof. From Eq.(101) we notice that the only element of the expression $\bigotimes_{j=1}^L e_j^{(\alpha_j \alpha_j)}$ in Eq.(53) which acts non trivially on the reference state $|i\rangle_{1\dots L}$ and its transpose ${}_{1\dots L} \langle i |$ is when $\alpha_1 = \alpha_2 = \dots = \alpha_L = i$. From our detailed analysis of Eq.(53) in Section (3) we know that this particular

set of α values in the sum of Eq.(57) only occurs when $\sigma = \mathcal{I}$ is the trivial permutation. Since $R_{1\dots L}^{\{\mathcal{I}\}} = \mathcal{I}_1 \otimes \dots \otimes \mathcal{I}_L$, this proves the first two relations in Eq.(102).

Focusing on the final two results, from the above analysis we find that the 1st, $\frac{3^L+1}{2}$ th and 3^L th row and column of $\mathcal{F}_{1\dots L}$ are given explicitly by 1 in the respective diagonal position, and zero everywhere else. Noting the following cofactor values concerning the 1st row and column of $\mathcal{F}_{1\dots L}$,

$$\begin{aligned} \det \{(\mathcal{F}_{1\dots L})_{kl}\}_{\substack{k=2\dots 3^L \\ l=2\dots 3^L}} &= \det \{\mathcal{F}_{1\dots L}\} \\ \det \{(\mathcal{F}_{1\dots L})_{kl}\}_{\substack{k=2\dots 3^L \\ l=1\dots (i-1)(i+1)\dots 3^L}} &= 0 \quad \text{for } i \in \{2, \dots, 3^L\} \\ \det \{(\mathcal{F}_{1\dots L})_{kl}\}_{\substack{k=1\dots (j-1)(j+1)\dots 3^L \\ l=2\dots 3^L}} &= 0 \quad \text{for } j \in \{2, \dots, 3^L\}, \end{aligned}$$

and similarly equivalent values for the $\frac{3^L+1}{2}$ th and 3^L th row and column of $\mathcal{F}_{1\dots L}$.

We now recall Cramer's rule for the inverse of a matrix,

$$(\mathcal{F}_{1\dots L}^{-1})_{ij} = \frac{(-1)^{i+j}}{\det \{\mathcal{F}_{1\dots L}\}} \det \{(\mathcal{F}_{1\dots L})_{kl}\}_{\substack{k=1\dots j-1, j+1\dots 3^L \\ l=1\dots i-1, i+1\dots 3^L}}. \quad (103)$$

Hence applying this equation we obtain,

$$\begin{aligned} (\mathcal{F}_{1\dots L}^{-1})_{11} &= 1 \\ (\mathcal{F}_{1\dots L}^{-1})_{i1} &= 0 \quad \text{for } i \in \{2, \dots, 3^L\} \\ (\mathcal{F}_{1\dots L}^{-1})_{1j} &= 0 \quad \text{for } j \in \{2, \dots, 3^L\}, \end{aligned}$$

and similarly equivalent results for the $\frac{3^L+1}{2}$ th and 3^L th row and column of $\mathcal{F}_{1\dots L}$ - thus verifying the last two results of this proposition. \square

We now start by computing the four possible basic non-trivial DWPF's consisting of the expectation values of tensor product of operators carrying a single charge index. These are the building blocks required to construct the domain wall partition functions involving mixed fields such as $C_{1\dots L}^{(1)}(\nu)$ and $C_{1\dots L}^{(2)}(\nu)$ or $B_{1\dots L}^{(1)}(\mu)$ and $B_{1\dots L}^{(2)}(\mu)$.

5.1 Single DWPF for $C^{(2)}$

We now consider our first basic DWPF, labeled $Z_L^{(C,2)}$, which is given explicitly by,

$$Z_L^{(C,2)}(\{\nu\}, \{\xi\}) = {}_{1\dots L} \langle 2 | C_{1\dots L}^{(2)}(\nu_L) \dots C_{1\dots L}^{(2)}(\nu_1) | 3 \rangle_{1\dots L}. \quad (104)$$

Applying the \mathcal{F} -matrix similarity transform to each of the $C^{(2)}$ operators in the above expression, and using the results of Eq.(102), Eq.(104) immediately becomes,

$$Z_L^{(C,2)}(\{\nu\}, \{\xi\}) = {}_{1\dots L}\langle 2|\tilde{C}_{1\dots L}^{(2)}(\nu_L) \dots \tilde{C}_{1\dots L}^{(2)}(\nu_1)|3\rangle_{1\dots L}. \quad (105)$$

We now are going to obtain a recurrence relation for the above expression. This is done by inserting a complete set of states between the operators $\tilde{C}_{1\dots L}^{(2)}(\nu_2)$ and $\tilde{C}_{1\dots L}^{(2)}(\nu_1)$ to obtain,

$$\begin{aligned} Z_L^{(C,2)}(\{\nu\}, \{\xi\}) &= \sum_{p=1}^L {}_{1\dots L}\langle 2|\tilde{C}_{1\dots L}^{(2)}(\nu_L) \dots \tilde{C}_{1\dots L}^{(2)}(\nu_2)|2\rangle_p |3\rangle_{\substack{1\dots L \\ \neq p}} \\ &\times {}_p\langle 2|_{\substack{1\dots L \\ \neq p}} \langle 3|\tilde{C}_{1\dots L}^{(2)}(\nu_1)|3\rangle_{1\dots L}. \end{aligned} \quad (106)$$

It is elementary to show that only the above terms of the complete set of states are non zero by considering the action of $e^{(23)}$ on the reference states. Through the property ${}_p\langle 2|_{\substack{1\dots L \\ \neq p}} \langle 3|e_l^{(23)} = {}_{1\dots L}\langle 3|\delta_{pl}$, one is able to derive the following identity,

$${}_p\langle 2|_{\substack{1\dots L \\ \neq p}} \langle 3|\tilde{C}_{1\dots L}^{(2)}(\nu_1)|3\rangle_{1\dots L} = c_{32}(\nu_1, \xi_p) \prod_{\substack{i=1 \\ \neq p}}^L a_3(\nu_1, \xi_i) \theta_3(\xi_i, \xi_p). \quad (107)$$

We now focus on the expression ${}_{1\dots L}\langle 2|\tilde{C}_{1\dots L}^{(2)}(\nu_L) \dots \tilde{C}_{1\dots L}^{(2)}(\nu_2)|2\rangle_p |3\rangle_{\substack{1\dots L \\ \neq p}}$. Using the fact that $e_p^{(23)}|2\rangle_p = 0$, we can discard the value p in the summations of the $\tilde{C}_{1\dots L}^{(2)}(\nu)$ operators. Hence applying this fact, and elementary matrix multiplication, the expression now becomes,

$$\begin{aligned} &{}_{1\dots L}\langle 2|\tilde{C}_{1\dots L}^{(2)}(\nu_L) \dots \tilde{C}_{1\dots L}^{(2)}(\nu_2)|2\rangle_p |3\rangle_{\substack{1\dots L \\ \neq p}} \\ &= \sum_{\substack{l_2, \dots, l_L=1 \\ l_2 \neq \dots \neq l_L \neq p}}^L \prod_{j=2}^L \frac{b_{32}(\nu_j, \xi_p)}{b_{32}(\xi_{l_j}, \xi_p) \theta_2(\xi_p, \xi_{l_j})} {}_{1\dots L}\langle 2|\tilde{C}_{1\dots L}^{(2, l_L)}(\nu_L) \dots \tilde{C}_{1\dots L}^{(2, l_2)}(\nu_2)|2\rangle_p |3\rangle_{\substack{1\dots L \\ \neq p}}, \end{aligned} \quad (108)$$

where,

$$\tilde{C}_{1\dots L}^{(2, l)}(\nu) = c_{32}(\nu, \xi_l) e_l^{(23)} \bigotimes_{\substack{i=1 \\ i \neq l, p}}^L \text{diag} \left\{ b_{21}(\nu, \xi_i), \frac{b_{32}(\nu, \xi_i)}{b_{32}(\xi_l, \xi_i) \theta_2(\xi_i, \xi_l)}, a_3(\nu, \xi_i) \theta_3(\xi_i, \xi_l) \right\}_i. \quad (109)$$

We now offer the following comments on Eq.(108). Firstly, the product $\prod_{j=2}^L \frac{b_{32}(\nu_j, \xi_p)}{b_{32}(\xi_{l_j}, \xi_p) \theta_2(\xi_p, \xi_{l_j})}$ is independent of the value of l_2, \dots, l_L due to the condition $l_2 \neq \dots \neq l_L \neq p$. Secondly, the operators $\tilde{C}_{1\dots L}^{(2, l)}(\nu)$ act trivially on the vector space V_p , meaning that we can decrease the number of

relevant vector spaces in the reference states by one. Thirdly, when the aforementioned coefficient is taken out of the sum over l_2, \dots, l_L , the following simplification occurs,

$$\begin{aligned} \sum_{\substack{l_2, \dots, l_L=1 \\ l_2 \neq \dots \neq l_L \neq p}}^{L} \langle 2 | \tilde{C}_{1 \dots L}^{(2, l_L)}(\nu_L) \dots \tilde{C}_{1 \dots L}^{(2, l_2)}(\nu_2) | 3 \rangle_{1 \dots L \neq p} &= \sum_{\substack{1 \dots L \\ \neq p}} \langle 2 | \tilde{C}_{1 \dots L}^{(2)}(\nu_L) \dots \tilde{C}_{1 \dots L}^{(2)}(\nu_2) | 3 \rangle_{1 \dots L \neq p} \\ &= Z_{L-1}^{(C, 2)}(\{\nu\}, \{\xi\} | \hat{\nu}_1, \hat{\xi}_p). \end{aligned}$$

Hence Eq.(106) becomes the following recurrence relation,

$$\begin{aligned} Z_L^{(C, 2)}(\{\nu\}, \{\xi\}) &= \sum_{p=1}^L c_{32}(\nu_1, \xi_p) \prod_{\substack{i=1 \\ \neq p}}^L \frac{a_3(\nu_1, \xi_i) \theta_3(\xi_i, \xi_p)}{b_{32}(\xi_i, \xi_p) \theta_2(\xi_p, \xi_i)} \prod_{j=2}^L b_{32}(\nu_j, \xi_p) \\ &\times Z_{L-1}^{(C, 2)}(\{\nu\}, \{\xi\} | \hat{\nu}_1, \hat{\xi}_p). \end{aligned} \quad (110)$$

5.1.1 Exact solution

We now can apply induction to verify the complete algebraic expression for the $C^{(2)}$ type DWPF in terms of Boltzmann weights. To begin for $L = 1$ and $L = 2$ we have,

$$\begin{aligned} Z_1^{(C, 2)}(\nu_1, \xi_1) &= c_{32}(\nu_1, \xi_1) \\ Z_2^{(C, 2)}(\{\nu\}, \{\xi\}) &= \frac{c_{32}(\nu_1, \xi_1) c_{32}(\nu_2, \xi_2) a_3(\nu_1, \xi_2) \theta_3(\xi_2, \xi_1) b_{32}(\nu_2, \xi_1)}{b_{32}(\xi_2, \xi_1) \theta_2(\xi_1, \xi_2)} \\ &+ \frac{c_{32}(\nu_1, \xi_2) c_{32}(\nu_1, \xi_2) a_3(\nu_1, \xi_1) \theta_3(\xi_1, \xi_2) b_{32}(\nu_2, \xi_2)}{b_{32}(\xi_1, \xi_2) \theta_2(\xi_2, \xi_1)} \end{aligned}$$

We now offer the following general result.

Proposition 3.

$$Z_L^{(C, 2)}(\{\nu\}, \{\xi\}) = \sum_{\sigma \in S_L} \prod_{i=1}^L c_{32}(\nu_i, \xi_{\sigma(i)}) \prod_{1 \leq j < k \leq L} \frac{a_3(\nu_j, \xi_{\sigma(k)}) b_{32}(\nu_k, \xi_{\sigma(j)}) \theta_3(\xi_{\sigma(k)}, \xi_{\sigma(j)})}{b_{32}(\xi_{\sigma(k)}, \xi_{\sigma(j)}) \theta_2(\xi_{\sigma(j)}, \xi_{\sigma(k)})} \quad (111)$$

Proof. Noting that the above formula is correct for $L = 1, 2$, we assume that it holds for some L , and focus on the $L + 1$ case of Eq.(110),

$$\begin{aligned} Z_{L+1}^{(C, 2)} &= \sum_{p=1}^{L+1} c_{32}(\nu_1, \xi_p) \prod_{\substack{m=1 \\ \neq p}}^{L+1} \frac{a_3(\nu_1, \xi_m) \theta_3(\xi_m, \xi_p)}{b_{32}(\xi_m, \xi_p) \theta_2(\xi_p, \xi_m)} \prod_{n=2}^{L+1} b_{32}(\nu_n, \xi_p) \\ &\times \sum_{\sigma \in S_L^{(p)}} \prod_{i=2}^{L+1} c_{32}(\nu_i, \xi_{\sigma(i)}) \prod_{2 \leq j < k \leq L+1} \frac{a_3(\nu_j, \xi_{\sigma(k)}) b_{32}(\nu_k, \xi_{\sigma(j)}) \theta_3(\xi_{\sigma(k)}, \xi_{\sigma(j)})}{b_{32}(\xi_{\sigma(k)}, \xi_{\sigma(j)}) \theta_2(\xi_{\sigma(j)}, \xi_{\sigma(k)})}, \end{aligned}$$

where the sum over the permutations with superscript p is given by,

$$\sum_{\sigma \in S_L^{(p)}} \equiv \sum_{\substack{\sigma_2=1 \\ \neq p}}^{L+1} \sum_{\substack{\sigma_3=1 \\ \neq p}}^{L+1} \dots \sum_{\substack{\sigma_{L+1}=1 \\ \neq p}}^{L+1} \quad \text{for } \sigma_2 \neq \sigma_3 \neq \dots \neq \sigma_{L+1}.$$

The verification of the proposition follows immediately through the change in label, $p \rightarrow \sigma_1$. \square

5.2 Single DWPF for $B^{(2)}$

We now consider the equivalent DWPF expression for the $B_{1...L}^{(2)}$ operators which is given explicitly by,

$$\begin{aligned} Z_L^{(B,2)}(\{\mu\}, \{\xi\}) &= {}_{1...L}\langle 3|B_{1...L}^{(2)}(\mu_1) \dots B_{1...L}^{(2)}(\mu_L)|2\rangle_{1...L} \\ &= {}_{1...L}\langle 3|\tilde{B}_{1...L}^{(2)}(\mu_1) \dots \tilde{B}_{1...L}^{(2)}(\mu_L)|2\rangle_{1...L}, \end{aligned} \quad (112)$$

where we have applied the results of Eq.(102) to twist the operators. As with the previous DWPF, we now obtain a recurrence relation for the above expression. We insert a complete set of states between the operators $\tilde{B}_{1...L}^{(2)}(\mu_1)$ and $\tilde{B}_{1...L}^{(2)}(\mu_2)$ to obtain,

$$\begin{aligned} Z_L^{(B,2)}(\{\mu\}, \{\xi\}) &= \sum_{p=1}^L {}_{1...L}\langle 3|\tilde{B}_{1...L}^{(2)}(\mu_1)|2\rangle_p |3\rangle_{\substack{1...L \\ \neq p}} \\ &\quad \times {}_p\langle 2|_{\substack{1...L \\ \neq p}} \langle 3|\tilde{B}_{1...L}^{(2)}(\mu_2) \dots \tilde{B}_{1...L}^{(2)}(\mu_L)|2\rangle_{1...L}. \end{aligned} \quad (113)$$

Once again one can show that only the above terms of the complete set of states are non zero by considering the action of $e^{(32)}$ on the reference states. Taking into account the property $e_l^{(32)}|2\rangle_p |3\rangle_{\substack{1...L \\ \neq p}} = \delta_{pl}|3\rangle_{1...L}$ one finds that,

$${}_{1...L}\langle 3|\tilde{B}_{1...L}^{(2)}(\mu_1)|2\rangle_p |3\rangle_{\substack{1...L \\ \neq p}} = c_{23}(\mu_1, \xi_p) \prod_{\substack{i=1 \\ \neq p}}^L \frac{a_3(\mu_1, \xi_i)}{b_{32}(\xi_i, \xi_p)\theta_3(\xi_p, \xi_i)}.$$

Let us now consider the expression ${}_p\langle 2|_{\substack{1...L \\ \neq p}} \langle 3|\tilde{B}_{1...L}^{(2)}(\mu_2) \dots \tilde{B}_{1...L}^{(2)}(\mu_L)|2\rangle_{1...L}$. Using the fact that ${}_p\langle 2|e_p^{(32)} = 0$, we can discard the value p in the summations of the $\tilde{B}_{1...L}^{(2)}(\mu)$ operators. Applying this fact, and elementary matrix multiplication, the expression becomes,

$$\begin{aligned} &{}_p\langle 2|_{\substack{1...L \\ \neq p}} \langle 3|\tilde{B}_{1...L}^{(2)}(\mu_2) \dots \tilde{B}_{1...L}^{(2)}(\mu_L)|2\rangle_{1...L} \\ &= \sum_{\substack{l_2, \dots, l_L=1 \\ l_2 \neq \dots \neq l_L \neq p}}^L \prod_{j=2}^L b_{32}(\mu_j, \xi_p)\theta_2(\xi_{l_j}, \xi_p) {}_p\langle 2|_{\substack{1...L \\ \neq p}} \langle 3|\tilde{B}_{1...L}^{(2, l_2)}(\mu_2) \dots \tilde{B}_{1...L}^{(2, l_L)}(\mu_L)|2\rangle_{1...L}, \end{aligned} \quad (114)$$

where,

$$\tilde{B}_{1...L}^{(2, l)}(\mu) = c_{23}(\mu, \xi_l) e_l^{(32)} \bigotimes_{\substack{i=1 \\ i \neq l, p}}^L \text{diag} \left\{ b_{31}(\mu, \xi_i), b_{32}(\mu, \xi_i)\theta_2(\xi_l, \xi_i), \frac{a_3(\mu, \xi_i)}{b_{32}(\xi_i, \xi_l)\theta_3(\xi_l, \xi_i)} \right\}_i.$$

As before the product $\prod_{j=2}^L b_{32}(\mu_j, \xi_p)\theta_2(\xi_{l_j}, \xi_p)$ is independent of the value of l_2, \dots, l_L and the operators $\tilde{B}_{1...L}^{(2, l)}(\mu)$ act trivially on the vector space V_p , meaning that we can decrease the number

of relevant vector spaces in the reference states by one. Applying these simplifications we find that,

$$\begin{aligned} \sum_{\substack{l_2, \dots, l_L=1 \\ l_2 \neq \dots \neq l_L \neq p}}^{1 \dots L} \langle 3 | \tilde{B}_{1 \dots L}^{(2, l_2)}(\mu_2) \dots \tilde{B}_{1 \dots L}^{(2, l_L)}(\mu_L) | 2 \rangle_{1 \dots L \neq p} &= \sum_{\substack{1 \dots L \\ \neq p}} \langle 3 | \tilde{B}_{1 \dots L}^{(2)}(\mu_2) \dots \tilde{B}_{1 \dots L}^{(2)}(\mu_L) | 2 \rangle_{1 \dots L \neq p} \\ &= Z_{L-1}^{(B, 2)}(\{\mu\}, \{\xi\} | \hat{\mu}_1, \hat{\xi}_p). \end{aligned}$$

Consequently, Eq.(113) becomes the following recurrence relation,

$$\begin{aligned} Z_L^{(B, 2)}(\{\mu\}, \{\xi\}) &= \sum_{p=1}^L c_{23}(\mu_1, \xi_p) \prod_{\substack{i=1 \\ \neq p}}^L \frac{a_3(\mu_1, \xi_i) \theta_2(\xi_i, \xi_p)}{b_{32}(\xi_i, \xi_p) \theta_3(\xi_p, \xi_i)} \prod_{j=2}^L b_{32}(\mu_j, \xi_p) \\ &\times Z_{L-1}^{(B, 2)}(\{\mu\}, \{\xi\} | \hat{\mu}_1, \hat{\xi}_p). \end{aligned} \quad (115)$$

5.2.1 Exact solution

We now offer the following general result,

$$Z_L^{(B, 2)}(\{\mu\}, \{\xi\}) = \sum_{\sigma \in S_L} \prod_{i=1}^L c_{23}(\mu_i, \xi_{\sigma(i)}) \prod_{1 \leq j < k \leq L} \frac{a_3(\mu_j, \xi_{\sigma(k)}) b_{32}(\mu_k, \xi_{\sigma(j)}) \theta_2(\xi_{\sigma(k)}, \xi_{\sigma(j)})}{b_{32}(\xi_{\sigma(k)}, \xi_{\sigma(j)}) \theta_3(\xi_{\sigma(j)}, \xi_{\sigma(k)})}, \quad (116)$$

where we note that the verification of Eq.(116) follows exactly from the verification of Eq.(111).

In what follows we shall consider the basic DWPF's that are constructed from considerably more complicated twisted operators. Nevertheless, the task of obtaining the explicit forms of the DWPF's of type $C^{(1)}$ and $B^{(1)}$ poses no greater challenge than what we have experienced thus far.

5.3 Single DWPF for $C^{(1)}$

The DWPF expression for the $C_{1 \dots L}^{(1)}$ operators is given explicitly by,

$$\begin{aligned} Z_L^{(C, 1)}(\{\nu\}, \{\xi\}) &= {}_{1 \dots L} \langle 1 | C_{1 \dots L}^{(1)}(\nu_L) \dots C_{1 \dots L}^{(1)}(\nu_1) | 3 \rangle_{1 \dots L} \\ &= {}_{1 \dots L} \langle 1 | \tilde{C}_{1 \dots L}^{(1)}(\nu_L) \dots \tilde{C}_{1 \dots L}^{(1)}(\nu_1) | 3 \rangle_{1 \dots L}, \end{aligned} \quad (117)$$

where we have applied the results of Eq.(102) to twist the operators.

Before inserting a complete set of states between operators we shall first consider the expression, $\tilde{C}_{1 \dots L}^{(1)}(\nu_1) | 3 \rangle_{1 \dots L}$, and in particular we notice that the action of the matrices $e_{l_1}^{(12)} \otimes e_{l_2}^{(23)}$ on the reference states $| 3 \rangle_{l_1} \otimes | 3 \rangle_{l_2}$ is zero. Hence if we insert a complete set of states in between the

operators $\tilde{C}_{1\dots L}^{(1)}(\nu_2)$ and $\tilde{C}_{1\dots L}^{(1)}(\nu_1)$ we obtain,

$$\begin{aligned} Z_L^{(C,1)}(\{\nu\}, \{\xi\}) &= \sum_{p=1}^L {}_{1\dots L}\langle 1|\tilde{C}_{1\dots L}^{(1)}(\nu_L) \dots \tilde{C}_{1\dots L}^{(1)}(\nu_2)|1\rangle_p|3\rangle_{\substack{1\dots L \\ \neq p}} \\ &\times {}_p\langle 1| {}_{1\dots L}\langle 3|\tilde{C}_{1\dots L}^{(1)}(\nu_1)|3\rangle_{1\dots L}, \end{aligned} \quad (118)$$

which is very similar to the corresponding expression for $\tilde{C}^{(2)}$.

We now use the identity ${}_p\langle 1| {}_{1\dots L}\langle 3|e_l^{(13)} = {}_{1\dots L}\langle 3|\delta_{pl}$ and by elementary matrix multiplication we obtain,

$${}_p\langle 1| {}_{1\dots L}\langle 3|\tilde{C}_{1\dots L}^{(1)}(\nu_1)|3\rangle_{1\dots L} = c_{31}(\nu_1, \xi_p) \prod_{\substack{i=1 \\ \neq p}}^L a_3(\nu_i, \xi_i) \theta_3(\xi_i, \xi_p).$$

Focusing on the expression ${}_{1\dots L}\langle 1|\tilde{C}_{1\dots L}^{(1)}(\nu_L) \dots \tilde{C}_{1\dots L}^{(1)}(\nu_2)|1\rangle_p|3\rangle_{\substack{1\dots L \\ \neq p}}$ we note that due to the elementary relations in Eq.(101), there is no possibility of the Wely matrices $e_{l_1}^{(12)} \otimes e_{l_2}^{(23)}$ in the expressions for the $\tilde{C}_{1\dots L}^{(1)}$ operators to produce anything but zero, hence they can be discarded from the calculations. Additionally, using the fact that $e_p^{(13)}|1\rangle_p = 0$, we can discard the value p in the summations of the $\tilde{C}_{1\dots L}^{(1)}(\nu)$ operators. Applying the above facts, and elementary matrix multiplication, the expression now becomes,

$$\begin{aligned} &{}_{1\dots L}\langle 1|\tilde{C}_{1\dots L}^{(1)}(\nu_L) \dots \tilde{C}_{1\dots L}^{(1)}(\nu_2)|1\rangle_p|3\rangle_{\substack{1\dots L \\ \neq p}} \\ &= \sum_{\substack{l_2, \dots, l_L=1 \\ l_2 \neq \dots \neq l_L \neq p}}^L \prod_{j=2}^L \frac{b_{21}(\nu_j, \xi_p)}{b_{21}(\xi_{l_j}, \xi_p) \theta_1(\xi_p, \xi_{l_j})} {}_{1\dots L}\langle 1|\tilde{C}_{1\dots L}^{(1, l_L)}(\nu_L) \dots \tilde{C}_{1\dots L}^{(1, l_2)}(\nu_2)|1\rangle_p|3\rangle_{\substack{1\dots L \\ \neq p}}, \end{aligned} \quad (119)$$

where,

$$\tilde{C}_{1\dots L}^{(1, l)}(\nu) = c_{31}(\nu, \xi_l) e_l^{(13)} \bigotimes_{\substack{i=1 \\ i \neq l, p}}^L \text{diag} \left\{ \frac{b_{21}(\nu, \xi_i)}{b_{21}(\xi_l, \xi_i) \theta_1(\xi_i, \xi_l)}, \frac{b_{32}(\nu, \xi_i)}{b_{32}(\xi_l, \xi_i)}, a_3(\nu, \xi_i) \theta_3(\xi_i, \xi_l) \right\}_i. \quad (120)$$

We note that the product $\prod_{j=2}^L \frac{b_{21}(\nu_j, \xi_p)}{b_{21}(\xi_{l_j}, \xi_p) \theta_1(\xi_p, \xi_{l_j})}$ is independent of the indices l_2, \dots, l_L , and the operators $\tilde{C}_{1\dots L}^{(1, l)}(\nu)$ act trivially on the vector space V_p , allowing us to decrease the number of relevant vector spaces in the reference states by one. Applying such simplifications one finds the relation,

$$\begin{aligned} \sum_{\substack{l_2, \dots, l_L=1 \\ l_2 \neq \dots \neq l_L \neq p}}^L {}_{1\dots L}\langle 1|\tilde{C}_{1\dots L}^{(1, l_L)}(\nu_L) \dots \tilde{C}_{1\dots L}^{(1, l_2)}(\nu_2)|3\rangle_{\substack{1\dots L \\ \neq p}} &= {}_{\substack{1\dots L \\ \neq p}}\langle 1|\tilde{C}_{1\dots L}^{(1)}(\nu_L) \dots \tilde{C}_{1\dots L}^{(1)}(\nu_2)|3\rangle_{\substack{1\dots L \\ \neq p}} \\ &= Z_{L-1}^{(C,1)}(\{\nu\}, \{\xi\}|\hat{\nu}_1, \hat{\xi}_p). \end{aligned}$$

Again, we can discard the terms involving $e_{l_1}^{(12)} \otimes e_{l_2}^{(23)}$ in the above $\tilde{C}_{1\dots L}^{(1)} \neq p$ operators because they always produce zero. As a consequence of that, Eq.(118) becomes the following recurrence relation,

$$Z_L^{(C,1)}(\{\nu\}, \{\xi\}) = \sum_{p=1}^L c_{31}(\nu_1, \xi_p) \prod_{\substack{i=1 \\ \neq p}}^L \frac{a_3(\nu_1, \xi_i) \theta_3(\xi_i, \xi_p)}{b_{21}(\xi_i, \xi_p) \theta_1(\xi_p, \xi_i)} \prod_{j=2}^L b_{21}(\nu_j, \xi_p) \times Z_{L-1}^{(C,1)}(\{\nu\}, \{\xi\} | \hat{\nu}_1, \hat{\xi}_p). \quad (121)$$

The same type of arguments presented in Section 5.1.1 can be used to provide the general solution to the above recurrence relation. It is given explicitly by,

$$Z_L^{(C,1)}(\{\nu\}, \{\xi\}) = \sum_{\sigma \in S_L} \prod_{i=1}^L c_{31}(\nu_i, \xi_{\sigma(i)}) \prod_{1 \leq j < k \leq L} \frac{a_3(\nu_j, \xi_{\sigma(k)}) b_{21}(\nu_k, \xi_{\sigma(j)}) \theta_3(\xi_{\sigma(k)}, \xi_{\sigma(j)})}{b_{21}(\xi_{\sigma(k)}, \xi_{\sigma(j)}) \theta_1(\xi_{\sigma(j)}, \xi_{\sigma(k)})}. \quad (122)$$

5.4 Single DWPF for type $B^{(1)}$

We now consider the equivalent DWPF expression for the $B_{1\dots L}^{(1)}$ operators which given explicitly by,

$$\begin{aligned} Z_L^{(B,1)}(\{\mu\}, \{\xi\}) &= {}_{1\dots L} \langle 3 | B_{1\dots L}^{(1)}(\mu_1) \dots B_{1\dots L}^{(1)}(\mu_L) | 1 \rangle_{1\dots L} \\ &= {}_{1\dots L} \langle 3 | \tilde{B}_{1\dots L}^{(1)}(\mu_1) \dots \tilde{B}_{1\dots L}^{(1)}(\mu_L) | 1 \rangle_{1\dots L}, \end{aligned} \quad (123)$$

where we have applied the results of Eq.(102) to twist the operators.

As with the case for $C^{(1)}$, before we insert a complete set of states between operators we shall first consider the expression ${}_{1\dots L} \langle 3 | \tilde{B}_{1\dots L}^{(1)}(\mu_1)$, and in particular we notice that the action of the matrices $e_{l_1}^{(21)} \otimes e_{l_2}^{(32)}$ on the transpose reference states ${}_{l_1} \langle 3 | \otimes {}_{l_2} \langle 3 |$ is zero. Hence if we insert a complete set of states in between the operators $\tilde{B}_{1\dots L}^{(1)}(\mu_1)$ and $\tilde{B}_{1\dots L}^{(1)}(\mu_2)$ we obtain,

$$\begin{aligned} Z_L^{(B,1)}(\{\mu\}, \{\xi\}) &= \sum_{p=1}^L {}_{1\dots L} \langle 3 | \tilde{B}_{1\dots L}^{(1)}(\mu_1) | 1 \rangle_p | 3 \rangle_{\substack{1\dots L \\ \neq p}} \\ &\times {}_p \langle 1 | {}_{1\dots L} \langle 3 | \tilde{B}_{1\dots L}^{(1)}(\mu_2) \dots \tilde{B}_{1\dots L}^{(1)}(\mu_L) | 1 \rangle_{1\dots L}, \end{aligned} \quad (124)$$

Using the fact that $e_l^{(31)} | 1 \rangle_p | 3 \rangle_{\substack{1\dots L \\ \neq p}} = \delta_{pl} | 3 \rangle_{1\dots L}$ one is bale to write the following identity,

$${}_{1\dots L} \langle 3 | \tilde{B}_{1\dots L}^{(1)}(\mu_1) | 1 \rangle_p | 3 \rangle_{\substack{1\dots L \\ \neq p}} = c_{13}(\mu_1, \xi_p) \prod_{\substack{i=1 \\ \neq p}}^L \frac{a_3(\mu_1, \xi_i)}{b_{31}(\xi_i, \xi_p) \theta_3(\xi_p, \xi_i)}.$$

In parallel to Section 5.3, due to the elementary relations in Eq.(101), there is no possibility of the Wely matrices $e_{l_1}^{(21)} \otimes e_{l_2}^{(32)}$ in the expression for the $\tilde{B}_{1\dots L}^{(1)}$ operators to produce anything

but zero, hence they can be discarded from the calculations. In addition, using the fact that ${}_p\langle 1|e_p^{(31)} = 0$, we can discard the value p in the summations of the $\tilde{B}_{1\dots L}^{(1)}$ operators. Considering these facts together with elementary matrix multiplication, the expression now becomes,

$$\begin{aligned} & {}_p\langle 1| \prod_{\substack{1\dots L \\ \neq p}} \langle 3|\tilde{B}_{1\dots L}^{(1)}(\mu_2) \dots \tilde{B}_{1\dots L}^{(1)}(\mu_L)|1\rangle_{1\dots L} \\ &= \sum_{\substack{l_2, \dots, l_L=1 \\ l_2 \neq \dots \neq l_L \neq p}}^L \prod_{j=2}^L b_{31}(\mu_j, \xi_p) \theta_1(\xi_{l_j}, \xi_p) {}_p\langle 1| \prod_{\substack{1\dots L \\ \neq p}} \langle 3|\tilde{B}_{1\dots L}^{(1, l_2)}(\mu_2) \dots \tilde{B}_{1\dots L}^{(1, l_L)}(\mu_L)|1\rangle_{1\dots L}, \end{aligned} \quad (125)$$

where,

$$\tilde{B}_{1\dots L}^{(1, l)}(\mu) = c_{13}(\mu, \xi_l) e_l^{(31)} \bigotimes_{\substack{i=1 \\ i \neq l, p}}^L \text{diag} \left\{ b_{31}(\mu, \xi_i) \theta_1(\xi_l, \xi_i), \frac{b_{32}(\mu, \xi_i)}{b_{21}(\xi_i, \xi_l)}, \frac{a_3(\mu, \xi_i)}{b_{31}(\xi_i, \xi_l) \theta_3(\xi_l, \xi_i)} \right\}. \quad (126)$$

Note that the product $\prod_{j=2}^L b_{31}(\mu_j, \xi_p) \theta_1(\xi_{l_j}, \xi_p)$ does not depend on the indices l_2, \dots, l_L and the operators $\tilde{B}_{1\dots L}^{(1, l)}$ act trivially on the vector space V_p , allowing us to decrease the number of relevant vector spaces in the reference states by one. These simplifications lead to,

$$\begin{aligned} \sum_{\substack{l_2, \dots, l_L=1 \\ l_2 \neq \dots \neq l_L \neq p}}^L \prod_{\substack{1\dots L \\ \neq p}} \langle 3|\tilde{B}_{1\dots L}^{(1, l_2)}(\mu_2) \dots \tilde{B}_{1\dots L}^{(1, l_L)}(\mu_L)|1\rangle_{1\dots L} &= \prod_{\substack{1\dots L \\ \neq p}} \langle 3|\tilde{B}_{1\dots L}^{(1)}(\mu_2) \dots \tilde{B}_{1\dots L}^{(1)}(\mu_L)|1\rangle_{1\dots L} \\ &= Z_{L-1}^{(B, 1)}(\{\mu\}, \{\xi\} | \hat{\mu}_1, \hat{\xi}_p). \end{aligned}$$

Again, we can discard the terms involving $e_{l_1}^{(21)} \otimes e_{l_2}^{(32)}$ in the above $\tilde{B}_{1\dots L}^{(1)}$ operators because they always produce zero. Hence Eq.(124) becomes the following recurrence relation,

$$\begin{aligned} Z_L^{(B, 1)}(\{\mu\}, \{\xi\}) &= \sum_{p=1}^L c_{13}(\mu_1, \xi_p) \prod_{\substack{i=1 \\ \neq p}}^L \frac{a_3(\mu_1, \xi_i) \theta_1(\xi_i, \xi_p)}{b_{31}(\xi_i, \xi_p) \theta_3(\xi_p, \xi_i)} \prod_{j=2}^L b_{31}(\mu_j, \xi_p) \\ &\times Z_{L-1}^{(B, 1)}(\{\mu\}, \{\xi\} | \hat{\mu}_1, \hat{\xi}_p). \end{aligned} \quad (127)$$

The exact solution to the above recurrence relation is,

$$Z_L^{(B, 1)}(\{\mu\}, \{\xi\}) = \sum_{\sigma \in S_L} \prod_{i=1}^L c_{13}(\mu_i, \xi_{\sigma(i)}) \prod_{1 \leq j < k \leq L} \frac{a_3(\mu_j, \xi_{\sigma(k)}) b_{31}(\mu_k, \xi_{\sigma(j)}) \theta_1(\xi_{\sigma(k)}, \xi_{\sigma(j)})}{b_{31}(\xi_{\sigma(k)}, \xi_{\sigma(j)}) \theta_3(\xi_{\sigma(j)}, \xi_{\sigma(k)})}. \quad (128)$$

6 Mixed DWPF

We start by discussing the preliminary steps to compute the DWPF's involving mixed operators. The elements of the Bethe state vectors built from C fields for the fifteen-vertex model are given by,

$$C_{1\dots L}^{(i_1)}(\nu_L) C_{1\dots L}^{(i_2)}(\nu_{L-1}) \dots C_{1\dots L}^{(i_L)}(\nu_1) |3\rangle_{1\dots L}, \quad (129)$$

where $i_1, \dots, i_L \in \{1, 2\}$. Here we are considering the vector $|3\rangle_{1\dots L}$ as our starting ferromagnetic reference state.

We now introduce the integer M , $M \leq L$, which indicates how many type “(1)” operators we have in our state vector element. Given M , every element of Eq.(129) will generally consist of an expression of $C^{(1)}$ and $C^{(2)}$ operators in no particular order. Applying the following Yang-Baxter algebra expression generated from Eq.(4),

$$C_{1\dots L}^{(1)}(\nu)C_{1\dots L}^{(2)}(\mu) = \frac{a_3}{b_{21}}(\mu, \nu)C_{1\dots L}^{(2)}(\mu)C_{1\dots L}^{(1)}(\nu) - \frac{c_{12}}{b_{21}}(\mu, \nu)C_{1\dots L}^{(2)}(\nu)C_{1\dots L}^{(1)}(\mu), \quad (130)$$

it is possible to commute all the $C^{(2)}$ operators to the left, leading to the general expression,

$$\begin{aligned} & C_{1\dots L}^{(i_1)}(\nu_L)C_{1\dots L}^{(i_2)}(\nu_{L-1}) \dots C_{1\dots L}^{(i_L)}(\nu_1)|3\rangle_{1\dots L} \\ = & \sum_{\sigma \in S_L} \phi_{M, \{\sigma\}}^{(i_1, \dots, i_L)} C_{1\dots L}^{(2)}(\nu_{\sigma(L)}) \dots C_{1\dots L}^{(2)}(\nu_{\sigma(M+1)}) C_{1\dots L}^{(1)}(\nu_{\sigma(M)}) \dots C_{1\dots L}^{(1)}(\nu_{\sigma(1)})|3\rangle_{1\dots L}, \end{aligned} \quad (131)$$

where the coefficient $\phi_{M, \{\sigma\}}^{(i_1, \dots, i_L)}$, which is constructed from the Boltzmann weights a_3 , b_{21} and c_{12} , generally depends on the initial value of the indices i_1, \dots, i_L , the integer M and the particular permutation σ .

From Eq.(131) we have that the most fundamental mixed DWPF expression of type C will be of the form,

$$Z_{L,M}^{(C)}(\{\nu\}, \{\xi\}) = \sum_{\substack{q_1 \dots q_M \\ \neq q_1 \dots q_M}} \langle 1 | \prod_{i=1}^L C_{1\dots L}^{(i)}(\nu_i) | 3 \rangle_{1\dots L}, \quad (132)$$

where for clarity we assume that $q_1 < q_2 < \dots < q_M$. We briefly note that the configuration in Figure 2 corresponds to Eq.(132) for $q_1 = 1, q_2 = 2, \dots, q_M = M$.

Similarly for B , the elements of the transpose Bethe state vectors are given by,

$${}_{1\dots L} \langle 3 | B_{1\dots L}^{(i_1)}(\mu_1) B_{1\dots L}^{(i_2)}(\mu_2) \dots B_{1\dots L}^{(i_L)}(\mu_L), \quad (133)$$

where again $i_1, \dots, i_L \in \{1, 2\}$.

Using M , $M \leq L$, to indicate how many “(1)” operators we have in our transpose state vector element, we apply the following Yang-Baxter algebra expression generated from Eq.(4),

$$B_{1\dots L}^{(2)}(\mu)B_{1\dots L}^{(1)}(\nu) = \frac{a_3}{b_{21}}(\mu, \nu)B_{1\dots L}^{(1)}(\nu)B_{1\dots L}^{(2)}(\mu) - \frac{c_{21}}{b_{21}}(\mu, \nu)B_{1\dots L}^{(1)}(\mu)B_{1\dots L}^{(2)}(\nu), \quad (134)$$

to commute all the $B^{(2)}$ operators to the right, leading to the expression,

$$\begin{aligned} & {}_{1\dots L}\langle 3|B_{1\dots L}^{(i_1)}(\mu_1)B_{1\dots L}^{(i_2)}(\mu_2)\dots B_{1\dots L}^{(i_L)}(\mu_L) \\ &= \sum_{\sigma \in S_L} \psi_{M,\{\sigma\}}^{(i_1,\dots,i_L)} {}_{1\dots L}\langle 3|B_{1\dots L}^{(1)}(\mu_{\sigma(1)})\dots B_{1\dots L}^{(1)}(\mu_{\sigma(M)})B_{1\dots L}^{(2)}(\mu_{\sigma(M+1)})\dots B_{1\dots L}^{(2)}(\mu_{\sigma(L)}), \end{aligned} \quad (135)$$

where the coefficient $\psi_{M,\{\sigma\}}^{(i_1,\dots,i_L)}$, is constructed from the Boltzmann weights a_3 , b_{21} and c_{21} and parallels $\phi_{M,\{\sigma\}}^{(i_1,\dots,i_L)}$ from Eq. (131) in structure.

From Eq.(135) we see that the most fundamental mixed DWPF expression of type B will be of the form,

$$Z_{L,M}^{(B)}(\{\mu\}, \{\xi\}) = {}_{1\dots L}\langle 3|B_{1\dots L}^{(1)}(\mu_1)\dots B_{1\dots L}^{(1)}(\mu_M)B_{1\dots L}^{(2)}(\mu_{M+1})\dots B_{1\dots L}^{(2)}(\mu_L)|1\rangle_{q_1\dots q_M}|2\rangle_{\substack{1\dots L \\ \neq q_1\dots q_M}}. \quad (136)$$

For clarity we briefly note that the configuration in Figure 3 corresponds to Eq.(136) for $q_1 = 1, q_2 = 2, \dots, q_M = M$.

We now devote separate subsections for the explicit evaluation of Eqs. (132) and (136).

6.1 Mixed DWPF for B

Applying the \mathcal{F} -matrix to Eq.(136) to twist the monodromy operators we obtain,

$$Z_{L,M}^{(B)}(\{\mu\}, \{\xi\}) = {}_{1\dots L}\langle 3|\tilde{B}_{1\dots L}^{(1)}(\mu_1)\dots \tilde{B}_{1\dots L}^{(1)}(\mu_M)\tilde{B}_{1\dots L}^{(2)}(\mu_{M+1})\dots \tilde{B}_{1\dots L}^{(2)}(\mu_L)\mathcal{F}_{1\dots L}|1\rangle_{q_1\dots q_M}|2\rangle_{\substack{1\dots L \\ \neq q_1\dots q_M}}. \quad (137)$$

We note that there is an $\mathcal{F}_{1\dots L}$ operator on the far right of Eq.(137) as there is no equivalent expression such as Eq.(102) for mixed reference states. We proceed by inserting two complete sets of states, but for clarity we shall do this in separate stages. Consider first placing a complete set of states in between the operators $\tilde{B}_{1\dots L}^{(1)}(\mu_M)$ and $\tilde{B}_{1\dots L}^{(2)}(\mu_{M+1})$ to obtain,

$$\begin{aligned} Z_{L,M}^{(B)}(\{\mu\}, \{\xi\}) &= \sum_{1 \leq p_1 < \dots < p_M \leq L} {}_{1\dots L}\langle 3|\tilde{B}_{1\dots L}^{(1)}(\mu_1)\dots \tilde{B}_{1\dots L}^{(1)}(\mu_M)|1\rangle_{p_1\dots p_M}|3\rangle_{\substack{1\dots L \\ \neq p_1\dots p_M}} \\ &\times {}_{p_1\dots p_M}\langle 1|{}_{\substack{1\dots L \\ \neq p_1\dots p_M}}\langle 3|\tilde{B}_{1\dots L}^{(2)}(\mu_{M+1})\dots \tilde{B}_{1\dots L}^{(2)}(\mu_L)\mathcal{F}_{1\dots L}|1\rangle_{q_1\dots q_M}|2\rangle_{\substack{1\dots L \\ \neq q_1\dots q_M}}. \end{aligned} \quad (138)$$

In order to verify that the above terms of the complete set of states are the only ones which are non zero, consider the expression,

$${}_{1\dots L}\langle 3|\tilde{B}_{1\dots L}^{(1)}(\mu_1)\dots \tilde{B}_{1\dots L}^{(1)}(\mu_M),$$

and recall from Section 5.4 that only the Weyl matrix e_l^{31} (as opposed to $e_{l_1}^{32} \otimes e_{l_2}^{21}$) gives non zero terms when applied to the reference state ${}_{1\dots L}\langle 3|$. Hence since ${}_l\langle 3|e_l^{31} = {}_l\langle 1|$, this means the only terms in the complete set of states which are non zero are,

$$\sum_{1 \leq p_1 < \dots < p_M \leq L} |1\rangle_{p_1\dots p_M} |3\rangle_{\substack{1\dots L \\ \neq p_1\dots p_M}} |1\rangle_{\substack{1\dots L \\ \neq p_1\dots p_M}} \langle 3|.$$

We now insert another complete set of states in between $\tilde{B}_{1\dots L}^{(2)}(\mu_L)$ and $\mathcal{F}_{1\dots L}$ in Eq.(138) to obtain,

$$\begin{aligned} Z_{L,M}^{(B)}(\{\mu\}, \{\xi\}) &= \sum_{1 \leq p_1 < \dots < p_M \leq L} {}_{1\dots L}\langle 3| \tilde{B}_{1\dots L}^{(1)}(\mu_1) \dots \tilde{B}_{1\dots L}^{(1)}(\mu_M) |1\rangle_{p_1\dots p_M} |3\rangle_{\substack{1\dots L \\ \neq p_1\dots p_M}} \\ &\times {}_{p_1\dots p_M}\langle 1| {}_{\substack{1\dots L \\ \neq p_1\dots p_M}} \langle 3| \tilde{B}_{1\dots L}^{(2)}(\mu_{M+1}) \dots \tilde{B}_{1\dots L}^{(2)}(\mu_L) |1\rangle_{p_1\dots p_M} |2\rangle_{\substack{1\dots L \\ \neq p_1\dots p_M}} \\ &\times {}_{p_1\dots p_M}\langle 1| {}_{\substack{1\dots L \\ \neq p_1\dots p_M}} \langle 2| \mathcal{F}_{1\dots L} |1\rangle_{q_1\dots q_M} |2\rangle_{\substack{1\dots L \\ \neq q_1\dots q_M}}. \end{aligned} \quad (139)$$

We now decrease the number of relevant vector spaces in each of the above *pseudo* basic DWPF expressions.

6.1.1 Reducing the number of relevant vector spaces - I

Beginning with ${}_{p_1\dots p_M}\langle 1| {}_{\substack{1\dots L \\ \neq p_1\dots p_M}} \langle 3| \tilde{B}_{1\dots L}^{(2)}(\mu_{M+1}) \dots \tilde{B}_{1\dots L}^{(2)}(\mu_L) |1\rangle_{p_1\dots p_M} |2\rangle_{\substack{1\dots L \\ \neq p_1\dots p_M}}$, we use elementary matrix algebra and the fact that ${}_p\langle 1|e_p^{(32)} = 0$ to discard the values p_1, \dots, p_M in the summations of the $\tilde{B}_{1\dots L}^{(2)}(\mu)$ operators to obtain,

$$\begin{aligned} &{}_{p_1\dots p_M}\langle 1| {}_{\substack{1\dots L \\ \neq p_1\dots p_M}} \langle 3| \tilde{B}_{1\dots L}^{(2)}(\mu_{M+1}) \dots \tilde{B}_{1\dots L}^{(2)}(\mu_L) |1\rangle_{p_1\dots p_M} |2\rangle_{\substack{1\dots L \\ \neq p_1\dots p_M}} \\ &= \prod_{i=M+1}^L \prod_{j=1}^M b_{31}(\mu_i, \xi_{p_j}) {}_{p_1\dots p_M}\langle 1| {}_{\substack{1\dots L \\ \neq p_1\dots p_M}} \langle 3| \tilde{B}_{1\dots L}^{(2)}(\mu_{M+1}) \dots \tilde{B}_{1\dots L}^{(2)}(\mu_L) |1\rangle_{p_1\dots p_M} |2\rangle_{\substack{1\dots L \\ \neq p_1\dots p_M}}. \end{aligned} \quad (140)$$

Since the operators $\tilde{B}_{1\dots L}^{(2)}(\mu)$ act trivially on the vector space V_{p_i} , $i = 1, \dots, M$ we decrease the number of relevant vector spaces in the reference states by M - hence Eq.(140) becomes,

$$\begin{aligned} &{}_{p_1\dots p_M}\langle 1| {}_{\substack{1\dots L \\ \neq p_1\dots p_M}} \langle 3| \tilde{B}_{1\dots L}^{(2)}(\mu_{M+1}) \dots \tilde{B}_{1\dots L}^{(2)}(\mu_L) |1\rangle_{p_1\dots p_M} |2\rangle_{\substack{1\dots L \\ \neq p_1\dots p_M}} \\ &= \prod_{i=M+1}^L \prod_{j=1}^M b_{31}(\mu_i, \xi_{p_j}) {}_{\substack{1\dots L \\ \neq p_1\dots p_M}} \langle 3| \tilde{B}_{1\dots L}^{(2)}(\mu_{M+1}) \dots \tilde{B}_{1\dots L}^{(2)}(\mu_L) |2\rangle_{\substack{1\dots L \\ \neq p_1\dots p_M}} \\ &= \prod_{i=M+1}^L \prod_{j=1}^M b_{31}(\mu_i, \xi_{p_j}) Z_{L-M}^{(B,2)}(\{\mu_k\}_{k=M+1,\dots,L} | \{\xi_l\}_{\substack{l=1,\dots,L \\ \neq p_1,\dots,p_M}}). \end{aligned}$$

6.1.2 Reducing the number of relevant vector spaces - II

We now focus on the term ${}_{1\dots L}\langle 3|\tilde{B}_{1\dots L}^{(1)}(\mu_1)\dots\tilde{B}_{1\dots L}^{(1)}(\mu_M)|1\rangle_{p_1\dots p_M}|3\rangle_{\substack{1\dots L \\ \neq p_1\dots p_M}}$. We reiterate that due to the elementary relations in Eq.(101), there is no possibility of the Weyl matrices $e_{l_1}^{(21)}\otimes e_{l_2}^{(32)}$ in the expressions for the $\tilde{B}_{1\dots L}^{(1)}$ operators producing anything but zero, hence they can be discarded from the calculations. Additionally, using the fact that $e_p^{(31)}|3\rangle_p = 0$, we can discard the values $l \neq p_1, \dots, p_M$ in the summations of the $\tilde{B}_{1\dots L}^{(1)}$ operators. Applying the above facts, and elementary matrix multiplication, the expression now becomes,

$$\begin{aligned} & {}_{1\dots L}\langle 3|\tilde{B}_{1\dots L}^{(1)}(\mu_1)\dots\tilde{B}_{1\dots L}^{(1)}(\mu_M)|1\rangle_{p_1\dots p_M}|3\rangle_{\substack{1\dots L \\ \neq p_1\dots p_M}} \\ = & \sum_{\substack{l_1,\dots,l_M=1 \\ l_1\neq\dots\neq l_M}}^M \prod_{i=1}^M \prod_{\substack{j=1 \\ \neq p_1,\dots,p_M}}^L \frac{a_3(\mu_i, \xi_j)}{b_{31}(\xi_j, \xi_{p_{l_i}})\theta_3(\xi_{p_{l_i}}, \xi_j)} \\ & \times {}_{1\dots L}\langle 3|\tilde{B}_{p_1\dots p_M}^{(1,p_{l_1})}(\mu_1)\dots\tilde{B}_{p_1\dots p_M}^{(1,p_{l_M})}(\mu_M)|1\rangle_{p_1\dots p_M}|3\rangle_{\substack{1\dots L \\ \neq p_1\dots p_M}}, \end{aligned} \quad (141)$$

where $\tilde{B}_{p_1\dots p_M}^{(1,p_j)}(\mu)$ is given by Eq. (126).

Since the product $\prod_{i=1}^M \prod_{\substack{j=1 \\ \neq p_1,\dots,p_M}}^L \frac{a_3(\mu_i, \xi_j)}{b_{31}(\xi_j, \xi_{p_{l_i}})\theta_3(\xi_{p_{l_i}}, \xi_j)}$ is independent of the value of l_1, \dots, l_M , and the operators $\tilde{B}_{p_1\dots p_M}^{(1,p_l)}(\mu)$ act trivially on the vector space V_i , $i \neq p_1, \dots, p_M$, we can decrease the number of relevant vector spaces in the reference states by $L - M$ and perform the following simplification to Eq.(141),

$$\begin{aligned} & {}_{1\dots L}\langle 3|\tilde{B}_{1\dots L}^{(1)}(\mu_1)\dots\tilde{B}_{1\dots L}^{(1)}(\mu_M)|1\rangle_{p_1\dots p_M}|3\rangle_{\substack{1\dots L \\ \neq p_1\dots p_M}} \\ = & \prod_{i=1}^M \prod_{\substack{j=1 \\ \neq p_1,\dots,p_M}}^L \frac{a_3(\mu_i, \xi_j)}{b_{31}(\xi_j, \xi_{p_i})\theta_3(\xi_{p_i}, \xi_j)} \sum_{\substack{l_1,\dots,l_M=1 \\ l_1\neq\dots\neq l_M}}^M {}_{p_1\dots p_M}\langle 3|\tilde{B}_{p_1\dots p_M}^{(1,p_{l_1})}(\mu_1)\dots\tilde{B}_{p_1\dots p_M}^{(1,p_{l_M})}(\mu_M)|1\rangle_{p_1\dots p_M} \\ = & \prod_{i=1}^M \prod_{\substack{j=1 \\ \neq p_1,\dots,p_M}}^L \frac{a_3(\mu_i, \xi_j)}{b_{31}(\xi_j, \xi_{p_i})\theta_3(\xi_{p_i}, \xi_j)} {}_{p_1\dots p_M}\langle 3|\tilde{B}_{p_1\dots p_M}^{(1)}(\mu_1)\dots\tilde{B}_{p_1\dots p_M}^{(1)}(\mu_M)|1\rangle_{p_1\dots p_M} \\ = & \prod_{i=1}^M \prod_{\substack{j=1 \\ \neq p_1,\dots,p_M}}^L \frac{a_3(\mu_i, \xi_j)}{b_{31}(\xi_j, \xi_{p_i})\theta_3(\xi_{p_i}, \xi_j)} Z_M^{(B,1)}(\{\mu_i\}_{i=1,\dots,M}|\{\xi_{p_j}\}_{j=1,\dots,M}). \end{aligned}$$

Applying all the results from this section we obtain the following form for the mixed DWPF of type B completely in terms of Boltzmann weights and the \mathcal{F} -matrix sandwiched between reference states,

$$\begin{aligned} Z_{L,M}^{(B)}(\{\mu\}, \{\xi\}) &= \sum_{1\leq p_1<\dots<p_M\leq L} {}_{p_1\dots p_M}\langle 1|_{\substack{1\dots L \\ \neq p_1\dots p_M}} \langle 2|\mathcal{F}_{1\dots L}|1\rangle_{q_1\dots q_M}|2\rangle_{\substack{1\dots L \\ \neq q_1\dots q_M}} \\ &\times \prod_{i=M+1}^L \prod_{j=1}^M b_{31}(\mu_i, \xi_{p_j}) \prod_{k=1}^M \prod_{\substack{l=1 \\ \neq p_1,\dots,p_M}}^L \frac{a_3(\mu_k, \xi_l)}{b_{31}(\xi_l, \xi_{p_k})\theta_3(\xi_{p_k}, \xi_l)} \\ &\times Z_M^{(B,1)}(\{\mu_m\}_{m=1,\dots,M}|\{\xi_{p_n}\}_{n=1,\dots,M}) Z_{L-M}^{(B,2)}(\{\mu_r\}_{r=M+1,\dots,L}|\{\xi_s\}_{\substack{s=1,\dots,L \\ \neq p_1,\dots,p_M}}). \end{aligned} \quad (142)$$

6.2 Mixed DWPF for C

We now apply the \mathcal{F} -matrix to Eq.(132) to twist the monodromy operators to obtain,

$$Z_{L,M}^{(C)}(\{\nu\}, \{\xi\}) = \sum_{q_1 \dots q_M} \langle 1 |_{\substack{1 \dots L \\ \neq q_1 \dots q_M}} \langle 2 | \mathcal{F}_{1 \dots L}^{-1} \tilde{C}_{1 \dots L}^{(2)}(\nu_L) \dots \tilde{C}_{1 \dots L}^{(2)}(\nu_{M+1}) \tilde{C}_{1 \dots L}^{(1)}(\nu_M) \dots \tilde{C}_{1 \dots L}^{(1)}(\nu_1) | 3 \rangle_{1 \dots L}. \quad (143)$$

Similarly to the previous mixed DWPF we note that there is an $\mathcal{F}_{1 \dots L}^{-1}$ operator on the far left of Eq.(143) as there is no equivalent expression such as Eq.(102) for mixed transpose reference states. As in the previous calculation for $Z_{L,M}^{(B)}$ we proceed by inserting two complete sets of states into Eq.(143) in separate stages for the sake of clarity. Consider first placing a complete set of states in between the operators $\tilde{C}_{1 \dots L}^{(2)}(\nu_{M+1})$ and $\tilde{C}_{1 \dots L}^{(1)}(\nu_M)$ to obtain,

$$\begin{aligned} Z_{L,M}^{(C)}(\{\nu\}, \{\xi\}) &= \sum_{1 \leq p_1 < \dots < p_M \leq L} \sum_{p_1 \dots p_M} \langle 1 |_{\substack{1 \dots L \\ \neq p_1 \dots p_M}} \langle 3 | \tilde{C}_{1 \dots L}^{(1)}(\nu_M) \dots \tilde{C}_{1 \dots L}^{(1)}(\nu_1) | 3 \rangle_{1 \dots L} \\ &\times \sum_{q_1 \dots q_M} \langle 1 |_{\substack{1 \dots L \\ \neq q_1 \dots q_M}} \langle 2 | \mathcal{F}_{1 \dots L}^{-1} \tilde{C}_{1 \dots L}^{(2)}(\nu_L) \dots \tilde{C}_{1 \dots L}^{(2)}(\nu_{M+1}) | 1 \rangle_{p_1 \dots p_M} | 3 \rangle_{\substack{1 \dots L \\ \neq p_1 \dots p_M}}. \end{aligned} \quad (144)$$

To verify that the above terms of the complete set of states are the only ones which are non zero, consider the expression,

$$\tilde{C}_{1 \dots L}^{(1)}(\nu_M) \dots \tilde{C}_{1 \dots L}^{(1)}(\nu_1) | 3 \rangle_{1 \dots L},$$

and recall from Section 5.3 that only the Weyl matrix e_l^{13} (as opposed to $e_{l_1}^{23} \otimes e_{l_2}^{12}$) gives non zero terms when applied to the reference state $|3\rangle_{1 \dots L}$. Hence since $e_l^{13}|3\rangle_l = |1\rangle_l$, this means the only terms in the complete set of states which are non zero are,

$$\sum_{1 \leq p_1 < \dots < p_M \leq L} |1\rangle_{p_1 \dots p_M} |3\rangle_{\substack{1 \dots L \\ \neq p_1 \dots p_M}} \sum_{p_1 \dots p_M} \langle 1 |_{\substack{1 \dots L \\ \neq p_1 \dots p_M}} \langle 3 |.$$

We now insert another complete set of states in between $\mathcal{F}_{1 \dots L}^{-1}$ and $\tilde{C}_{1 \dots L}^{(2)}(\nu_L)$ in Eq.(144) to obtain,

$$\begin{aligned} Z_{L,M}^{(C)}(\{\nu\}, \{\xi\}) &= \sum_{1 \leq p_1 < \dots < p_M \leq L} \sum_{q_1 \dots q_M} \langle 1 |_{\substack{1 \dots L \\ \neq q_1 \dots q_M}} \langle 2 | \mathcal{F}_{1 \dots L}^{-1} | 1 \rangle_{p_1 \dots p_M} | 2 \rangle_{\substack{1 \dots L \\ \neq p_1 \dots p_M}} \\ &\times \sum_{p_1 \dots p_M} \langle 1 |_{\substack{1 \dots L \\ \neq p_1 \dots p_M}} \langle 3 | \tilde{C}_{1 \dots L}^{(1)}(\nu_M) \dots \tilde{C}_{1 \dots L}^{(1)}(\nu_1) | 3 \rangle_{1 \dots L} \\ &\times \sum_{p_1 \dots p_M} \langle 1 |_{\substack{1 \dots L \\ \neq p_1 \dots p_M}} \langle 2 | \tilde{C}_{1 \dots L}^{(2)}(\nu_L) \dots \tilde{C}_{1 \dots L}^{(2)}(\nu_{M+1}) | 1 \rangle_{p_1 \dots p_M} | 3 \rangle_{\substack{1 \dots L \\ \neq p_1 \dots p_M}}, \end{aligned} \quad (145)$$

and proceed to decrease the number of relevant vector spaces in each of the above pseudo basic DWPF expressions.

6.2.1 Reducing the number of relevant vector spaces - I

Beginning with ${}_{p_1 \dots p_M} \langle 1 | \prod_{\substack{1 \dots L \\ \neq p_1 \dots p_M}} \langle 2 | \tilde{C}_{1 \dots L}^{(2)}(\nu_L) \dots \tilde{C}_{1 \dots L}^{(2)}(\nu_{M+1}) | 1 \rangle_{p_1 \dots p_M} | 3 \rangle \prod_{\substack{1 \dots L \\ \neq p_1 \dots p_M}}$, we use the fact that $e_p^{(23)} | 1 \rangle_p = 0$ to discard the values (p_1, \dots, p_M) in the summations of the $\tilde{C}_{1 \dots L}^{(2)}(\nu)$ operators. Hence applying this fact, and elementary matrix multiplication, the expression simplifies to,

$$\begin{aligned} & {}_{p_1 \dots p_M} \langle 1 | \prod_{\substack{1 \dots L \\ \neq p_1 \dots p_M}} \langle 2 | \tilde{C}_{1 \dots L}^{(2)}(\nu_L) \dots \tilde{C}_{1 \dots L}^{(2)}(\nu_{M+1}) | 1 \rangle_{p_1 \dots p_M} | 3 \rangle \prod_{\substack{1 \dots L \\ \neq p_1 \dots p_M}} \\ = & \prod_{i=M+1}^L \prod_{j=1}^M b_{21}(\nu_i, \xi_{p_j}) {}_{p_1 \dots p_M} \langle 1 | \prod_{\substack{1 \dots L \\ \neq p_1 \dots p_M}} \langle 2 | \tilde{C}_{1 \dots L}^{(2)}(\nu_L) \dots \tilde{C}_{1 \dots L}^{(2)}(\nu_{M+1}) | 1 \rangle_{p_1 \dots p_M} | 3 \rangle \prod_{\substack{1 \dots L \\ \neq p_1 \dots p_M}}. \end{aligned} \quad (146)$$

Since the operators $\tilde{C}_{1 \dots L}^{(2)}(\nu)$ act trivially on the vector space V_{p_i} , $i = 1, \dots, M$, we can decrease the number of relevant vector spaces in the reference states by M - hence Eq.(146) becomes,

$$\begin{aligned} & {}_{p_1 \dots p_M} \langle 1 | \prod_{\substack{1 \dots L \\ \neq p_1 \dots p_M}} \langle 2 | \tilde{C}_{1 \dots L}^{(2)}(\nu_L) \dots \tilde{C}_{1 \dots L}^{(2)}(\nu_{M+1}) | 1 \rangle_{p_1 \dots p_M} | 3 \rangle \prod_{\substack{1 \dots L \\ \neq p_1 \dots p_M}} \\ = & \prod_{i=M+1}^L \prod_{j=1}^M b_{21}(\nu_i, \xi_{p_j}) \prod_{\substack{1 \dots L \\ \neq p_1 \dots p_M}} \langle 2 | \tilde{C}_{1 \dots L}^{(2)}(\nu_L) \dots \tilde{C}_{1 \dots L}^{(2)}(\nu_{M+1}) | 3 \rangle \prod_{\substack{1 \dots L \\ \neq p_1 \dots p_M}} \\ = & \prod_{i=M+1}^L \prod_{j=1}^M b_{21}(\nu_i, \xi_{p_j}) Z_{L-M}^{(C,2)}(\{\nu_k\}_{k=M+1, \dots, L} | \{\xi_l\}_{l=1, \dots, L}^{\substack{l=1, \dots, L \\ \neq p_1, \dots, p_M}}). \end{aligned}$$

6.2.2 Reducing the number of relevant vector spaces - II

We now focus on the term ${}_{p_1 \dots p_M} \langle 1 | \prod_{\substack{1 \dots L \\ \neq p_1 \dots p_M}} \langle 3 | \tilde{C}_{1 \dots L}^{(1)}(\nu_M) \dots \tilde{C}_{1 \dots L}^{(1)}(\nu_1) | 3 \rangle_{1 \dots L}$. Recall that due to the elementary relations in Eq.(101), there is no possibility of the Weyl matrices $e_{l_1}^{(12)} \otimes e_{l_2}^{(23)}$ in the expressions for the $\tilde{C}_{1 \dots L}^{(1)}$ operators to produce anything but zero, hence they can be discarded from the calculations. Additionally, using the fact that ${}_p \langle 3 | e_p^{(13)} = 0$, we can discard the values $l \neq p_1, \dots, p_M$ in the summations of the $\tilde{C}_{1 \dots L}^{(1)}$ operators. Applying the above facts, and elementary matrix multiplication, the expression now becomes,

$$\begin{aligned} & {}_{p_1 \dots p_M} \langle 1 | \prod_{\substack{1 \dots L \\ \neq p_1 \dots p_M}} \langle 3 | \tilde{C}_{1 \dots L}^{(1)}(\nu_M) \dots \tilde{C}_{1 \dots L}^{(1)}(\nu_1) | 3 \rangle_{1 \dots L} \\ = & \sum_{\substack{l_1, \dots, l_M=1 \\ l_1 \neq \dots \neq l_M}}^M \prod_{i=1}^M \prod_{\substack{j=1 \\ \neq p_1, \dots, p_M}}^L a_3(\nu_i, \xi_j) \theta_3(\xi_j, \xi_{p_{l_i}}) \\ & \times {}_{p_1 \dots p_M} \langle 1 | \prod_{\substack{1 \dots L \\ \neq p_1 \dots p_M}} \langle 3 | \tilde{C}_{p_1 \dots p_M}^{(1, p_M)}(\nu_M) \dots \tilde{C}_{p_1 \dots p_M}^{(1, p_1)}(\nu_1) | 3 \rangle_{1 \dots L}, \end{aligned} \quad (147)$$

where $\tilde{C}_{p_1 \dots p_M}^{(1, p_j)}(\nu)$ is given by Eq. (120).

Since the product $\prod_{i=1}^M \prod_{\substack{j=1 \\ \neq p_1, \dots, p_M}}^L a_3(\nu_i, \xi_j) \theta_3(\xi_j, \xi_{p_{l_i}})$ is independent of the value of l_1, \dots, l_M , and the operators $\tilde{C}_{p_1 \dots p_M}^{(1, p_i)}(\nu)$ act trivially on the vector spaces V_i , $i \neq p_1, \dots, p_M$, we can decrease

the number of relevant vector spaces in the reference states by $L - M$ and simplify Eq. (147) as follows,

$$\begin{aligned}
& {}_{p_1 \dots p_M} \langle 1 | \prod_{\substack{1 \dots L \\ \neq p_1 \dots p_M}} \langle 3 | \tilde{C}_{1 \dots L}^{(1)}(\nu_M) \dots \tilde{C}_{1 \dots L}^{(1)}(\nu_1) | 3 \rangle_{1 \dots L} \\
&= \prod_{i=1}^M \prod_{\substack{j=1 \\ \neq p_1, \dots, p_M}}^L a_3(\nu_i, \xi_j) \theta_3(\xi_j, \xi_{p_i}) \sum_{\substack{l_1, \dots, l_M=1 \\ l_1 \neq \dots \neq l_M}}^M {}_{p_1 \dots p_M} \langle 1 | \tilde{C}_{p_1 \dots p_M}^{(1, p_{l_M})}(\nu_M) \dots \tilde{C}_{p_1 \dots p_M}^{(1, p_{l_1})}(\nu_1) | 3 \rangle_{p_1 \dots p_M} \\
&= \prod_{i=1}^M \prod_{\substack{j=1 \\ \neq p_1, \dots, p_M}}^L a_3(\nu_i, \xi_j) \theta_3(\xi_j, \xi_{p_i}) {}_{p_1 \dots p_M} \langle 1 | \tilde{C}_{p_1 \dots p_M}^{(1)}(\nu_M) \dots \tilde{C}_{p_1 \dots p_M}^{(1)}(\nu_1) | 3 \rangle_{p_1 \dots p_M} \\
&= \prod_{i=1}^M \prod_{\substack{j=1 \\ \neq p_1, \dots, p_M}}^L a_3(\nu_i, \xi_j) \theta_3(\xi_j, \xi_{p_i}) Z_M^{(C,1)}(\{\nu_i\}_{i=1, \dots, M} | \{\xi_{p_j}\}_{j=1, \dots, M}).
\end{aligned}$$

Hence we obtain the following form for the mixed DWPF of type C completely in terms of Boltzmann weights and the inverse \mathcal{F} -matrix sandwiched between reference states,

$$\begin{aligned}
Z_{L,M}^{(C)}(\{\nu\}, \{\xi\}) &= \sum_{1 \leq p_1 < \dots < p_M \leq L} {}_{q_1 \dots q_M} \langle 1 | \prod_{\substack{1 \dots L \\ \neq q_1 \dots q_M}} \langle 2 | \mathcal{F}_{1 \dots L}^{-1} | 1 \rangle_{p_1 \dots p_M} | 2 \rangle_{\substack{1 \dots L \\ \neq p_1 \dots p_M}} \\
&\quad \times \prod_{i=M+1}^L \prod_{j=1}^M b_{21}(\nu_i, \xi_{p_j}) \prod_{k=1}^M \prod_{\substack{l=1 \\ \neq p_1, \dots, p_M}}^L a_3(\nu_k, \xi_l) \theta_3(\xi_l, \xi_{p_k}) \\
&\quad \times Z_M^{(C,1)}(\{\nu_m\}_{m=1, \dots, M} | \{\xi_{p_n}\}_{n=1, \dots, M}) Z_{L-M}^{(C,2)}(\{\nu_r\}_{r=M+1, \dots, L} | \{\xi_s\}_{\substack{s=1, \dots, L \\ \neq p_1, \dots, p_M}}).
\end{aligned} \tag{148}$$

7 Conclusions

In this article we have argued that the factorized F -matrices associated to the R -matrix of the $U(1)^{(N-1)}$ vertex model can be constructed for arbitrary Boltzmann weights. Our analysis is purely algebraic relying only on the structure of the R -matrix as well as on the corresponding unitarity and Yang-Baxter relations. We have applied this formulation to the $N = 3$ fifteen-vertex model which is the simplest extension of the asymmetric $N = 2$ six-vertex model in such family of integrable models. For $N = 3$ we have exhibited the algebraic expressions of relevant monodromy matrix elements in the F -basis. This allowed us to compute the domain wall partition functions related to the creation and annihilation fields for arbitrary weights.

We have motivated our approach by showing that the Yang-Baxter relations for $N = 3$ hide a general structure of Boltzmann weights. The underlying algebraic variety is at least governed by the intersection of two quadrics in the projective \mathbb{P}^4 space leading us to a surface of del Pezzo type. Interestingly enough, this type of variety also governs the integrability of the $N = 2$ vertex model.

In fact, one can show that the weights of the asymmetric six-vertex model lie on a cubic del Pezzo surface. A natural question to investigate is whether the del Pezzo structure persists for general N or even higher dimension algebraic variety emerges when $N > 3$. In any case, this observation emphasizes the importance of attempts to establish results for integrable vertex models that are independent of any specific parameterization of Boltzmann weights.

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Appendix A: The Yang-Baxter for $N = 3$

Here we describe some details entering the general solution of the Yang-Baxter equation for $N = 3$ exhibited in Section 2. The algebraic solution consists in the elimination of the weights dependent on the variables ξ_1 and ξ_2 leading to determine the algebraic invariants constraining the remaining Boltzmann weights. We start by solving the relations involving only two triple products, see Eqs.(11-13). After eliminating the weights $c_{21}(\xi_1, \xi_2)$, $c_{32}(\xi_1, \xi_2)$, $b_{31}(\xi_1, \xi_2)$, $b_{13}(\xi_1, \xi_2)$, $c_{13}(\xi_1, \xi_2)$ and $b_{23}(\xi_1, \xi_2)$ one finds that there exists only three independent relations. They are easily separable providing us the following invariants,

$$\frac{b_{32}(\xi_i, \xi_3)}{b_{12}(\xi_i, \xi_3)} = \delta_1, \quad \frac{b_{31}(\xi_i, \xi_3)}{b_{21}(\xi_i, \xi_3)} = \delta_2, \quad \frac{b_{13}(\xi_i, \xi_3)}{b_{23}(\xi_i, \xi_3)} = \delta_3 \quad \text{for } i = 1, 2, \quad (\text{A.1})$$

where δ_1 , δ_2 and δ_3 are free parameters.

Taking into account this solution the number of relations with three triple products (14-21) reduce to thirty independent functional equations. Among of them there exists eight relations which are suitable for carrying out further elimination of weights. Their explicit forms are,

$$\begin{aligned} c_{12}(\xi_1, \xi_2)a_1(\xi_1, \xi_3)b_{21}(\xi_2, \xi_3) &+ b_{21}(\xi_1, \xi_2)c_{12}(\xi_1, \xi_3)c_{21}(\xi_2, \xi_3) \\ &= c_{12}(\xi_1, \xi_2)b_{21}(\xi_1, \xi_3)a_1(\xi_2, \xi_3), \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned}
b_{21}(\xi_1, \xi_2)c_{12}(\xi_1, \xi_3)b_{12}(\xi_2, \xi_3) &+ c_{12}(\xi_1, \xi_2)a_1(\xi_1, \xi_3)c_{12}(\xi_2, \xi_3) \\
&= a_1(\xi_1, \xi_2)c_{12}(\xi_1, \xi_3)a_1(\xi_2, \xi_3),
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
b_{12}(\xi_1, \xi_2)a_1(\xi_1, \xi_3)c_{21}(\xi_2, \xi_3) &+ c_{12}(\xi_1, \xi_2)c_{21}(\xi_1, \xi_3)b_{12}(\xi_2, \xi_3) \\
&= a_1(\xi_1, \xi_2)b_{12}(\xi_1, \xi_3)c_{21}(\xi_2, \xi_3),
\end{aligned} \tag{A.4}$$

$$\begin{aligned}
c_{12}(\xi_1, \xi_2)a_2(\xi_1, \xi_3)b_{12}(\xi_2, \xi_3) &+ b_{12}(\xi_1, \xi_2)c_{12}(\xi_1, \xi_3)c_{21}(\xi_2, \xi_3) \\
&= c_{12}(\xi_1, \xi_2)b_{12}(\xi_1, \xi_3)a_2(\xi_2, \xi_3),
\end{aligned} \tag{A.5}$$

$$\begin{aligned}
c_{23}(\xi_1, \xi_2)a_3(\xi_1, \xi_3)b_{21}(\xi_1, \xi_3)b_{23}(\xi_2, \xi_3) &+ b_{21}(\xi_1, \xi_2)c_{23}(\xi_1, \xi_3)b_{23}(\xi_1, \xi_3)c_{32}(\xi_2, \xi_3) \\
&= c_{23}(\xi_1, \xi_2)b_{23}(\xi_1, \xi_3)b_{21}(\xi_1, \xi_3)a_3(\xi_2, \xi_3),
\end{aligned} \tag{A.6}$$

$$\begin{aligned}
\delta_1 b_{21}(\xi_1, \xi_2)c_{23}(\xi_1, \xi_3)b_{23}(\xi_1, \xi_3)b_{12}(\xi_2, \xi_3) &+ c_{23}(\xi_1, \xi_2)a_3(\xi_1, \xi_3)b_{21}(\xi_1, \xi_3)c_{23}(\xi_2, \xi_3) \\
&= a_3(\xi_1, \xi_2)c_{23}(\xi_1, \xi_3)b_{21}(\xi_1, \xi_3)a_3(\xi_2, \xi_3),
\end{aligned} \tag{A.7}$$

$$\begin{aligned}
\delta_1 c_{23}(\xi_1, \xi_2)c_{32}(\xi_1, \xi_3)b_{12}(\xi_2, \xi_3) &+ b_{32}(\xi_1, \xi_2)a_3(\xi_1, \xi_3)c_{32}(\xi_2, \xi_3) \\
&= \delta_1 a_3(\xi_1, \xi_2)b_{12}(\xi_1, \xi_3)c_{32}(\xi_2, \xi_3),
\end{aligned} \tag{A.8}$$

$$\begin{aligned}
\delta_1 c_{23}(\xi_1, \xi_2)a_2(\xi_1, \xi_3)b_{12}(\xi_2, \xi_3) &+ b_{32}(\xi_1, \xi_2)c_{23}(\xi_1, \xi_3)c_{32}(\xi_2, \xi_3) \\
&= \delta_1 c_{23}(\xi_1, \xi_2)b_{12}(\xi_1, \xi_3)a_2(\xi_2, \xi_3).
\end{aligned} \tag{A.9}$$

From Eqs.(A.2-A.4) we are able to eliminate the weights $b_{21}(\xi_1, \xi_2)$, $a_1(\xi_1, \xi_2)$, $b_{12}(\xi_1, \xi_2)$ and by substituting them in Eq.(A.5) we find that it leads to the following constraint,

$$\frac{a_1(\xi_i, \xi_3)a_2(\xi_i, \xi_3) + b_{12}(\xi_i, \xi_3)b_{21}(\xi_i, \xi_3) - c_{12}(\xi_i, \xi_3)c_{21}(\xi_i, \xi_3)}{a_1(\xi_i, \xi_3)b_{12}(\xi_i, \xi_3)} = \delta_4 \quad \text{for } i = 1, 2, \tag{A.10}$$

where δ_4 is a constant.

The same procedure can be implemented for Eqs.(A.6-A.9). By eliminating the weights $a_3(\xi_1, \xi_2)$, $b_{32}(\xi_1, \xi_2)$ and $c_{23}(\xi_1, \xi_2)$ one obtains the additional constraint,

$$\frac{a_3(\xi_i, \xi_3)a_2(\xi_i, \xi_3) + \delta_1 b_{12}(\xi_i, \xi_3)b_{23}(\xi_i, \xi_3) - c_{23}(\xi_i, \xi_3)c_{32}(\xi_i, \xi_3)}{a_3(\xi_i, \xi_3)b_{12}(\xi_i, \xi_3)} = \delta_5 \quad \text{for } i = 1, 2, \tag{A.11}$$

where δ_5 is a free parameter.

At this point we are left to eliminate only the Boltzmann weights $c_{13}(\xi_1, \xi_2)$ and $a_2(\xi_1, \xi_2)$. This is done by performing linear combinations among certain remaining three terms relations coming from Eqs.(14-21). Remarkably enough all the consistency conditions are solved by means of the following extra invariants,

$$\frac{\delta_1 a_1(\xi_i, \xi_3) b_{23}(\xi_i, \xi_3) - a_3(\xi_i, \xi_3) b_{21}(\xi_i, \xi_3)}{b_{23}(\xi_i, \xi_3) b_{21}(\xi_i, \xi_3)} = \delta_6 \quad \text{for } i = 1, 2, \quad (\text{A.12})$$

$$\begin{aligned} & \frac{\delta_1 [a_1(\xi_1, \xi_3) c_{23}(\xi_1, \xi_3) - c_{13}(\xi_1, \xi_3) c_{21}(\xi_1, \xi_3)]}{c_{23}(\xi_1, \xi_3) b_{21}(\xi_1, \xi_3)} = \\ & \frac{\delta_1 a_1(\xi_2, \xi_3) c_{12}(\xi_2, \xi_3) b_{23}(\xi_2, \xi_3) - c_{13}(\xi_2, \xi_3) b_{21}(\xi_2, \xi_3) c_{32}(\xi_2, \xi_3)}{c_{12}(\xi_2, \xi_3) b_{23}(\xi_2, \xi_3) b_{21}(\xi_2, \xi_3)} = \delta_7, \end{aligned} \quad (\text{A.13})$$

$$\begin{aligned} & \frac{\delta_1 a_1(\xi_1, \xi_3) c_{12}(\xi_1, \xi_3) b_{23}(\xi_1, \xi_3) - c_{13}(\xi_1, \xi_3) b_{21}(\xi_1, \xi_3) c_{32}(\xi_1, \xi_3)}{c_{12}(\xi_1, \xi_3) b_{23}(\xi_1, \xi_3) b_{21}(\xi_1, \xi_3)} = \\ & \frac{\delta_1 [a_1(\xi_2, \xi_3) c_{23}(\xi_2, \xi_3) - c_{13}(\xi_2, \xi_3) c_{21}(\xi_2, \xi_3)]}{c_{23}(\xi_2, \xi_3) b_{21}(\xi_2, \xi_3)} = \delta_8, \end{aligned} \quad (\text{A.14})$$

$$\begin{aligned} & \frac{\delta_1 [a_1(\xi_1, \xi_3) c_{32}(\xi_1, \xi_3) - c_{12}(\xi_1, \xi_3) c_{31}(\xi_1, \xi_3)]}{b_{21}(\xi_1, \xi_3) c_{32}(\xi_1, \xi_3)} = \\ & \frac{\delta_1 a_1(\xi_2, \xi_3) b_{23}(\xi_2, \xi_3) c_{21}(\xi_2, \xi_3) - c_{23}(\xi_2, \xi_3) b_{21}(\xi_2, \xi_3) c_{31}(\xi_2, \xi_3)}{b_{23}(\xi_2, \xi_3) b_{21}(\xi_2, \xi_3) c_{21}(\xi_2, \xi_3)} = \delta_9, \end{aligned} \quad (\text{A.15})$$

$$\begin{aligned} & \frac{\delta_1 a_1(\xi_1, \xi_3) b_{23}(\xi_1, \xi_3) c_{21}(\xi_1, \xi_3) - c_{23}(\xi_1, \xi_3) b_{21}(\xi_1, \xi_3) c_{31}(\xi_1, \xi_3)}{b_{23}(\xi_1, \xi_3) b_{21}(\xi_1, \xi_3) c_{21}(\xi_1, \xi_3)} = \\ & \frac{\delta_1 [a_1(\xi_2, \xi_3) c_{32}(\xi_2, \xi_3) - c_{12}(\xi_2, \xi_3) c_{31}(\xi_2, \xi_3)]}{b_{21}(\xi_2, \xi_3) c_{32}(\xi_2, \xi_3)} = \delta_{10}, \end{aligned} \quad (\text{A.16})$$

where $\delta_6, \delta_7, \delta_8, \delta_9$ and δ_{10} are yet new free parameters.

It turns out that the remaining Yang-Baxter functional relations lead us to branches that impose further constraints among certain weights and the invariants obtained so far. We find that one such possible branch is,

$$\frac{a_2(\xi_i, \xi_3) b_{21}(\xi_i, \xi_3)}{a_1(\xi_i, \xi_3) b_{12}(\xi_i, \xi_3)} = \delta_1 \frac{a_2(\xi_i, \xi_3) b_{23}(\xi_i, \xi_3)}{a_3(\xi_i, \xi_3) b_{12}(\xi_i, \xi_3)} = \frac{\delta_1 (\delta_4 \delta_7 - \delta_1)}{\delta_7^2} \quad \text{for } i = 1, 2, \quad (\text{A.17})$$

while the invariants $\delta_3, \delta_5, \delta_6, \delta_8, \delta_9, \delta_{10}$ are fixed by,

$$\delta_3 = \frac{\delta_7^2}{\delta_2(\delta_4\delta_7 - \delta_1)}, \quad \delta_5 = \delta_4, \quad \delta_6 = 0, \quad \delta_8 = \delta_7, \quad \delta_9 = \delta_{10} = \frac{\delta_1\delta_7}{\delta_4\delta_7 - \delta_1}. \quad (\text{A.18})$$

We have now reached a point where all the weights in the variables ξ_1 and ξ_2 have been eliminated, while the weights in the variables $\xi_{1,2}$ and ξ_3 are constrained by the algebraic invariants (A.1,A.10-A.17). The final step of our analysis consists to make the intersection of these algebraic invariants. The procedure for performing such intersection is as follows. We first note that the weights $a_2(\xi_i, \xi_3)$, $b_{23}(\xi_i, \xi_3)$, $b_{31}(\xi_i, \xi_3)$, $b_{32}(\xi_i, \xi_3)$, $a_3(\xi_i, \xi_3)$, $c_{13}(\xi_i, \xi_3)$ and $c_{32}(\xi_i, \xi_3)$ can be linearly extracted from (A.1,A.11-A.17), leading us to,

$$a_2(\xi_i, \xi_3) = \frac{\delta_1(\delta_1 - \delta_4\delta_7)a_1(\xi_i, \xi_3)c_{12}(\xi_i, \xi_3)c_{21}(\xi_i, \xi_3)}{[\delta_1a_1(\xi_i, \xi_3) - \delta_7b_{21}(\xi_i, \xi_3)][(\delta_1 - \delta_4\delta_7)a_1(\xi_i, \xi_3) + \delta_7b_{21}(\xi_i, \xi_3)]}, \quad (\text{A.19})$$

$$b_{23}(\xi_i, \xi_3) = \frac{(\delta_1 - \delta_4\delta_7)b_{21}(\xi_i, \xi_3)c_{23}(\xi_i, \xi_3)c_{31}(\xi_i, \xi_3)}{c_{21}(\xi_i, \xi_3)\delta_1[(\delta_1 - \delta_4\delta_7)a_1(\xi_i, \xi_3) + \delta_7b_{21}(\xi_i, \xi_3)]}, \quad (\text{A.20})$$

$$b_{31}(\xi_i, \xi_3) = \delta_2b_{21}(\xi_i, \xi_3), \quad (\text{A.21})$$

$$b_{32}(\xi_i, \xi_3) = \frac{\delta_1\delta_7^2b_{21}(\xi_i, \xi_3)c_{12}(\xi_i, \xi_3)c_{21}(\xi_i, \xi_3)}{[\delta_1a_1(\xi_i, \xi_3) - \delta_7b_{21}(\xi_i, \xi_3)][(-\delta_1 + \delta_4\delta_7)a_1(\xi_i, \xi_3) - \delta_7b_{21}(\xi_i, \xi_3)]}, \quad (\text{A.22})$$

$$a_3(\xi_i, \xi_3) = \frac{(\delta_1 - \delta_4\delta_7)a_1(\xi_i, \xi_3)c_{23}(\xi_i, \xi_3)c_{31}(\xi_i, \xi_3)}{c_{21}(\xi_i, \xi_3)[(\delta_1 - \delta_4\delta_7)a_1(\xi_i, \xi_3) + \delta_7b_{21}(\xi_i, \xi_3)]}, \quad (\text{A.23})$$

$$c_{13}(\xi_i, \xi_3) = \frac{c_{23}(\xi_i, \xi_3)[\delta_1a_1(\xi_i, \xi_3) - \delta_7b_{21}(\xi_i, \xi_3)]}{\delta_1c_{21}(\xi_i, \xi_3)}, \quad (\text{A.24})$$

$$c_{32}(\xi_i, \xi_3) = \frac{(\delta_1 - \delta_4\delta_7)c_{12}(\xi_i, \xi_3)c_{31}(\xi_i, \xi_3)}{(\delta_1 - \delta_4\delta_7)a_1(\xi_i, \xi_3) + \delta_7b_{21}(\xi_i, \xi_3)}, \quad (\text{A.25})$$

where $i = 1, 2$.

We next substitute the weight (A.19) in Eq.(A.10) and as result we find that the weights $a_1(\xi_i, \xi_3)$, $b_{12}(\xi_i, \xi_3)$, $b_{21}(\xi_i, \xi_3)$, $c_{12}(\xi_i, \xi_3)$ and $c_{21}(\xi_i, \xi_3)$ are constrained by,

$$\left[\frac{\delta_1(\delta_4\delta_7 - \delta_1)}{\delta_7^2} \right] a_1(\xi_i, \xi_3)^2 b_{12}(\xi_i, \xi_3) - \delta_4 a_1(\xi_i, \xi_3) b_{12}(\xi_i, \xi_3) b_{21}(\xi_i, \xi_3) + b_{12}(\xi_i, \xi_3) b_{21}(\xi_i, \xi_3)^2 - b_{21}(\xi_i, \xi_3) c_{12}(\xi_i, \xi_3) c_{21}(\xi_i, \xi_3) = 0, \quad \text{for } i = 1, 2. \quad (\text{A.26})$$

By performing the definition $\delta_7 = \delta_1\Delta_1$ and $\delta_4 = \Delta_2$ we see that the form of Eq.(A.26) is the same as that of the hypersurface (23) given in the main text. This is the case because the spectral

parameter ξ_3 is a common variable for all the weights entering Eq.(A.26). Therefore, through the identification,

$$\begin{aligned} a_2(\xi_i, \xi_3) &= a(\xi_i), \quad b_{12}(\xi_i, \xi_3) = b(\xi_i), \quad b_{21}(\xi_i, \xi_3) = \bar{b}(\xi_i), \quad c_{12}(\xi_i, \xi_3) = c(\xi_i), \quad c_{21}(\xi_i, \xi_3) = \bar{c}(\xi_i), \\ c_{23}(\xi_i, \xi_3) &= h_1(\xi_i), \quad c_{31}(\xi_i, \xi_3) = h_2(\xi_i), \end{aligned} \quad (\text{A.27})$$

we see that Eq.(A.26) becomes exactly Eq.(23).

We conclude by observing that the earlier $U(1) \otimes U(1)$ Yang-Baxter solution presented in the literature [20, 21] is indeed a particular case of the R -matrix given in the text. In fact, the so called Perk-Schultz solution associated to the $U_q[Su(3)]$ quantum algebra is obtained by setting,

$$\begin{aligned} \delta_1 &= \delta_2 = 1, \quad \Delta_1 = q, \quad \Delta_4 = q + 1/q, \quad a(\xi) = 1, \quad \bar{b}(\xi) = b(\xi) = q \frac{\xi^2 - 1}{\xi^2 - q^2}, \\ h_1(\xi) &= \xi h_2(\xi) = \bar{c}(\xi) = c(\xi) = \xi \frac{q^2 - 1}{q^2 - \xi^2}, \end{aligned} \quad (\text{A.28})$$

where q is a free constant and ξ is the spectral parameter. In this special case we see that the R -matrix is of the difference form, i.e., $R = R(\xi_1/\xi_2)$.

Appendix B: The analysis for $L = 2$

Here we present the explicit expressions of the entries (91,92) together with the corresponding simplifications using the Yang-Baxter and unitarity relations;

$$\begin{aligned} \kappa_1^{(D)} &= c_{21}(\xi_1, \xi_2) \underbrace{\{b_{31}(\mu, \xi_1)b_{32}(\mu, \xi_2) - b_{32}(\mu, \xi_1)b_{31}(\mu, \xi_2)\}}_{\text{apply Eq.(12)-}\{3,1,2\}} = 0 \\ \kappa_2^{(D)} &= \underbrace{b_{31}(\mu, \xi_1)a_3(\mu, \xi_2)c_{31}(\xi_1, \xi_2) + c_{13}(\mu, \xi_1)c_{31}(\mu, \xi_2)b_{31}(\xi_1, \xi_2)}_{\text{apply Eq.(14)-}\{3,1\}} \\ &\quad - a_3(\mu, \xi_1)b_{31}(\mu, \xi_2)c_{31}(\xi_1, \xi_2) = 0 \\ \kappa_3^{(D)} &= \underbrace{b_{32}(\mu, \xi_1)a_3(\mu, \xi_2)c_{32}(\xi_1, \xi_2) + c_{23}(\mu, \xi_1)c_{32}(\mu, \xi_2)b_{32}(\xi_1, \xi_2)}_{\text{apply Eq.(14)-}\{3,2\}} \\ &\quad - a_3(\mu, \xi_1)b_{32}(\mu, \xi_2)c_{32}(\xi_1, \xi_2) = 0 \end{aligned}$$

$$\begin{aligned}
\kappa_1^{(C_2)} &= b_{21}(\mu, \xi_1) c_{32}(\mu, \xi_2) \\
\kappa_2^{(C_2)} &= a_2(\mu, \xi_1) c_{32}(\mu, \xi_2) - \frac{c_{32}(\mu, \xi_1) b_{32}(\mu, \xi_2) \overbrace{c_{32}(\xi_1, \xi_2)}^{\text{apply Eq. (10)-}\{3,2\}}}{b_{32}(\xi_1, \xi_2)} \\
&= \frac{1}{b_{32}(\xi_2, \xi_1)} \underbrace{\{c_{32}(\mu, \xi_2) a_2(\mu, \xi_1) b_{32}(\xi_2, \xi_1) + b_{32}(\mu, \xi_1) c_{32}(\mu, \xi_1) c_{23}(\xi_2, \xi_1)\}}_{\text{apply Eq. (19)-}\{3,2\}} \\
&= \frac{c_{32}(\mu, \xi_2) b_{32}(\mu, \xi_1) a_2(\xi_2, \xi_1)}{b_{32}(\xi_2, \xi_1)} \\
\kappa_3^{(C_2)} &= \underbrace{c_{32}(\mu, \xi_1) a_3(\mu, \xi_2) c_{32}(\xi_1, \xi_2) + b_{23}(\mu, \xi_1) c_{32}(\mu, \xi_2) b_{32}(\xi_1, \xi_2)}_{\text{apply Eq. (16)-}\{3,2\}} \\
&= a_3(\mu, \xi_1) c_{32}(\mu, \xi_2) a_3(\xi_1, \xi_2) \\
\kappa_4^{(C_2)} &= \frac{c_{32}(\mu, \xi_1) b_{31}(\mu, \xi_2) b_{21}(\xi_1, \xi_2)}{b_{31}(\xi_1, \xi_2)} \\
\kappa_5^{(C_2)} &= \frac{c_{32}(\mu, \xi_1) b_{32}(\mu, \xi_2)}{b_{32}(\xi_1, \xi_2)} \\
\kappa_6^{(C_2)} &= c_{32}(\mu, \xi_1) a_3(\mu, \xi_2) \\
\kappa_7^{(C_2)} &= b_{21}(\mu, \xi_1) c_{32}(\mu, \xi_2) c_{21}(\xi_1, \xi_2) + c_{12}(\mu, \xi_1) c_{31}(\mu, \xi_2) b_{21}(\xi_1, \xi_2) \\
&\quad \underbrace{\quad}_{\text{apply Eq. (13)-}\{2,3,1\}} \\
&\quad - \frac{c_{32}(\mu, \xi_1) \overbrace{b_{31}(\mu, \xi_2) b_{21}(\xi_1, \xi_2)}^{\text{apply Eq. (13)-}\{2,3,1\}} c_{31}(\xi_1, \xi_2)}{b_{31}(\xi_1, \xi_2)} \\
&= \underbrace{b_{21}(\mu, \xi_1) c_{32}(\mu, \xi_2) c_{21}(\xi_1, \xi_2) + c_{12}(\mu, \xi_1) c_{31}(\mu, \xi_2) b_{21}(\xi_1, \xi_2)}_{\text{apply Eq. (21)-}\{3,2,1\}} \\
&\quad - c_{32}(\mu, \xi_1) b_{21}(\mu, \xi_2) c_{31}(\xi_1, \xi_2) = 0
\end{aligned}$$

$$\begin{aligned}
\kappa_1^{(B_2)} &= b_{31}(\mu, \xi_1) c_{23}(\mu, \xi_2) \\
\kappa_2^{(B_2)} &= b_{32}(\mu, \xi_1) c_{23}(\mu, \xi_2) \\
\kappa_3^{(B_2)} &= \frac{a_3(\mu, \xi_1) c_{23}(\mu, \xi_2)}{b_{32}(\xi_1, \xi_2)} \\
\kappa_4^{(B_2)} &= \frac{c_{23}(\mu, \xi_1) b_{21}(\mu, \xi_2) b_{31}(\xi_1, \xi_2)}{b_{21}(\xi_1, \xi_2)} \\
\kappa_5^{(B_2)} &= \underbrace{c_{23}(\mu, \xi_1) a_2(\mu, \xi_2) b_{32}(\xi_1, \xi_2) + b_{32}(\mu, \xi_1) c_{23}(\mu, \xi_2) c_{32}(\xi_1, \xi_2)}_{\text{apply Eq.(18)-}\{2,3\}} \\
&= c_{23}(\mu, \xi_1) b_{32}(\mu, \xi_2) a_2(\xi_1, \xi_2) \\
&\quad \underbrace{\text{apply Eq.(10)-}\{2,3\}} \\
\kappa_6^{(B_2)} &= c_{23}(\mu, \xi_1) b_{23}(\mu, \xi_2) - \frac{a_3(\mu, \xi_1) c_{23}(\mu, \xi_2) \overbrace{c_{32}(\xi_1, \xi_2)}}{b_{32}(\xi_1, \xi_2)} \\
&= \frac{1}{b_{32}(\xi_2, \xi_1)} \underbrace{\{b_{23}(\mu, \xi_2) c_{23}(\mu, \xi_1) b_{32}(\xi_2, \xi_1) + c_{23}(\mu, \xi_2) a_3(\mu, \xi_1) c_{23}(\xi_2, \xi_1)\}}_{\text{apply Eq.(17)-}\{2,3\}} \\
&= \frac{a_3(\mu, \xi_2) c_{23}(\mu, \xi_1) a_3(\xi_2, \xi_1)}{b_{32}(\xi_2, \xi_1)} \\
\kappa_7^{(B_2)} &= c_{13}(\mu, \xi_1) c_{21}(\mu, \xi_2) b_{31}(\xi_1, \xi_2) + b_{31}(\mu, \xi_1) c_{23}(\mu, \xi_2) c_{31}(\xi_1, \xi_2) \\
&\quad \underbrace{\text{apply Eq.(13)-}\{3,2,1\}} \\
&\quad - \frac{c_{23}(\mu, \xi_1) \overbrace{b_{21}(\mu, \xi_2) b_{31}(\xi_1, \xi_2)} c_{21}(\xi_1, \xi_2)}{b_{21}(\xi_1, \xi_2)} \\
&= \underbrace{c_{13}(\mu, \xi_1) c_{21}(\mu, \xi_2) b_{31}(\xi_1, \xi_2) + b_{31}(\mu, \xi_1) c_{23}(\mu, \xi_2) c_{31}(\xi_1, \xi_2)}_{\text{apply Eq.(21)-}\{2,3,1\}} \\
&\quad - c_{23}(\mu, \xi_1) b_{31}(\mu, \xi_2) c_{21}(\xi_1, \xi_2) = 0
\end{aligned}$$

$$\begin{aligned}
\kappa_1^{(C_1)} &= a_1(\mu, \xi_1)c_{31}(\mu, \xi_2) - \frac{c_{31}(\mu, \xi_1)b_{31}(\mu, \xi_2)}{b_{31}(\xi_1, \xi_2)} \overbrace{c_{31}(\xi_1, \xi_2)}^{\text{apply Eq.(10)-}\{3,1\}} \\
&= \frac{1}{b_{31}(\xi_2, \xi_1)} \underbrace{\{c_{31}(\mu, \xi_2)a_1(\mu, \xi_1)b_{31}(\xi_2, \xi_1) + b_{31}(\mu, \xi_2)c_{31}(\mu, \xi_1)c_{13}(\xi_2, \xi_1)\}}_{\text{apply Eq.(19)-}\{3,1\}} \\
&= \frac{c_{31}(\mu, \xi_2)b_{31}(\mu, \xi_1)a_1(\xi_2, \xi_1)}{b_{31}(\xi_2, \xi_1)} \\
\kappa_2^{(C_1)} &= c_{21}(\mu, \xi_1)c_{32}(\mu, \xi_2)c_{21}(\xi_1, \xi_2) + b_{12}(\mu, \xi_1)c_{31}(\mu, \xi_2)b_{21}(\xi_1, \xi_2) \\
&\quad - \frac{c_{31}(\mu, \xi_1)b_{32}(\mu, \xi_2)c_{21}(\xi_1, \xi_2)}{b_{32}(\xi_1, \xi_2)} \overbrace{c_{32}(\xi_1, \xi_2)}^{\text{apply Eq.(10)-}\{3,2\}} \\
&= \frac{c_{21}(\xi_1, \xi_2)}{b_{32}(\xi_2, \xi_1)} \underbrace{\{c_{32}(\mu, \xi_2)c_{21}(\mu, \xi_1)b_{32}(\xi_2, \xi_1) + b_{32}(\mu, \xi_2)c_{31}(\mu, \xi_1)c_{23}(\xi_2, \xi_1)\}}_{\text{apply Eq.(20)-}\{3,2,1\}} \\
&\quad + b_{12}(\mu, \xi_1)c_{31}(\mu, \xi_2)b_{21}(\xi_1, \xi_2) \\
&= \frac{c_{31}(\mu, \xi_2)}{b_{32}(\xi_2, \xi_1)} \underbrace{\{c_{21}(\xi_1, \xi_2)c_{21}(\xi_2, \xi_1)b_{32}(\mu, \xi_1) + b_{12}(\mu, \xi_1)b_{21}(\xi_1, \xi_2)b_{32}(\xi_2, \xi_1)\}}_{\text{apply Eq.(9)-}\{2,1\}} \\
&= \frac{c_{31}(\mu, \xi_2)}{b_{32}(\xi_2, \xi_1)} \{b_{32}(\mu, \xi_1) + b_{21}(\xi_1, \xi_2) \underbrace{[b_{12}(\mu, \xi_1)b_{32}(\xi_2, \xi_1) - b_{32}(\mu, \xi_1)b_{12}(\xi_2, \xi_1)]}_{\text{apply Eq.(13)-}\{1,3,2\}}\} \\
&= \frac{b_{32}(\mu, \xi_1)c_{31}(\mu, \xi_2)}{b_{32}(\xi_2, \xi_1)} \\
\kappa_3^{(C_1)} &= \underbrace{c_{31}(\mu, \xi_1)a_3(\mu, \xi_2)c_{31}(\xi_1, \xi_2) + b_{13}(\mu, \xi_1)c_{31}(\mu, \xi_2)b_{31}(\xi_1, \xi_2)}_{\text{apply Eq.(16)-}\{3,1\}} \\
&= a_3(\mu, \xi_1)c_{31}(\mu, \xi_2)a_3(\xi_1, \xi_2) \\
\kappa_4^{(C_1)} &= \frac{c_{31}(\mu, \xi_1)b_{31}(\mu, \xi_2)}{b_{31}(\xi_1, \xi_2)} \\
\kappa_5^{(C_1)} &= \frac{c_{31}(\mu, \xi_1)b_{32}(\mu, \xi_2)}{b_{32}(\xi_1, \xi_2)} \\
\kappa_6^{(C_1)} &= c_{31}(\mu, \xi_1)a_3(\mu, \xi_2) \\
\kappa_7^{(C_1)} &= c_{21}(\mu, \xi_1)c_{32}(\mu, \xi_2) - \frac{c_{31}(\mu, \xi_1)b_{32}(\mu, \xi_2)}{b_{32}(\xi_1, \xi_2)} \overbrace{c_{32}(\xi_1, \xi_2)}^{\text{apply Eq.(10)-}\{3,2\}} \\
&= \frac{1}{b_{32}(\xi_2, \xi_1)} \underbrace{\{c_{32}(\mu, \xi_2)c_{21}(\mu, \xi_1)b_{32}(\xi_2, \xi_1) + b_{32}(\mu, \xi_2)c_{31}(\mu, \xi_1)c_{23}(\xi_2, \xi_1)\}}_{\text{apply Eq.(20)-}\{3,2,1\}} \\
&= \frac{c_{31}(\mu, \xi_2)b_{32}(\mu, \xi_1)c_{21}(\xi_2, \xi_1)}{b_{32}(\xi_2, \xi_1)} \\
\kappa_8^{(C_1)} &= \frac{c_{31}(\mu, \xi_1)b_{32}(\mu, \xi_2)c_{21}(\xi_1, \xi_2)}{b_{32}(\xi_1, \xi_2)}
\end{aligned}$$

$$\begin{aligned}
\kappa_1^{(B_1)} &= b_{31}(\mu, \xi_1) c_{13}(\mu, \xi_2) \\
\kappa_2^{(B_1)} &= \frac{b_{32}(\mu, \xi_1) c_{13}(\mu, \xi_2)}{b_{21}(\xi_1, \xi_2)} \\
\kappa_3^{(B_1)} &= \frac{a_3(\mu, \xi_1) c_{13}(\mu, \xi_2)}{b_{31}(\xi_1, \xi_2)} \\
\kappa_4^{(B_1)} &= \underbrace{c_{13}(\mu, \xi_1) a_1(\mu, \xi_2) b_{31}(\xi_1, \xi_2) + b_{31}(\mu, \xi_1) c_{13}(\mu, \xi_2) c_{31}(\xi_1, \xi_2)}_{\text{apply Eq.(18)-}\{1,3\}} \\
&= c_{13}(\mu, \xi_1) b_{31}(\mu, \xi_2) a_1(\xi_1, \xi_2) \\
\kappa_5^{(B_1)} &= - \underbrace{\frac{c_{21}(\xi_1, \xi_2)}{b_{21}(\xi_1, \xi_2)}}_{\text{apply Eq.(10)-}\{2,1\}} \underbrace{\{c_{23}(\mu, \xi_1) c_{12}(\mu, \xi_2) b_{32}(\xi_1, \xi_2) + b_{32}(\mu, \xi_1) c_{13}(\mu, \xi_2) c_{32}(\xi_1, \xi_2)\}}_{\text{apply Eq.(21)-}\{1,3,2\}} \\
&\quad + c_{13}(\mu, \xi_1) b_{12}(\mu, \xi_2) b_{32}(\xi_1, \xi_2) \\
&= \frac{c_{13}(\mu, \xi_1)}{b_{21}(\xi_2, \xi_1)} \underbrace{\{c_{12}(\xi_1, \xi_2) c_{12}(\xi_2, \xi_1) b_{32}(\mu, \xi_2) + b_{12}(\mu, \xi_2) b_{32}(\xi_1, \xi_2) b_{21}(\xi_2, \xi_1)\}}_{\text{apply Eq.(9)-}\{1,2\}} \\
&= \frac{c_{13}(\mu, \xi_1)}{b_{21}(\xi_2, \xi_1)} \{b_{32}(\mu, \xi_2) + b_{21}(\xi_2, \xi_1) \underbrace{[b_{12}(\mu, \xi_2) b_{32}(\xi_1, \xi_2) - b_{32}(\mu, \xi_2) b_{12}(\xi_1, \xi_2)]}_{\text{apply Eq.(13)-}\{1,3,2\}}\} \\
&= \frac{c_{12}(\mu, \xi_1) b_{32}(\mu, \xi_2)}{b_{21}(\xi_2, \xi_1)} \underbrace{\hspace{10em}}_{\text{apply Eq.(10)-}\{3,1\}} \\
\kappa_6^{(B_1)} &= c_{13}(\mu, \xi_1) b_{13}(\mu, \xi_2) - \frac{a_3(\mu, \xi_1) c_{13}(\mu, \xi_2)}{b_{31}(\xi_1, \xi_2)} \underbrace{c_{31}(\xi_1, \xi_2)}_{\text{apply Eq.(10)-}\{3,1\}} \\
&= \frac{1}{b_{31}(\xi_2, \xi_1)} \underbrace{\{b_{13}(\mu, \xi_2) c_{13}(\mu, \xi_1) b_{31}(\xi_2, \xi_1) + c_{13}(\mu, \xi_2) a_3(\mu, \xi_1) c_{13}(\xi_2, \xi_1)\}}_{\text{apply Eq.(17)-}\{1,3\}} \\
&= \frac{a_3(\mu, \xi_2) c_{13}(\mu, \xi_1) a_3(\xi_2, \xi_1)}{b_{31}(\xi_2, \xi_1)} \underbrace{\hspace{10em}}_{\text{apply Eq.(10)-}\{2,1\}} \\
\kappa_7^{(B_1)} &= - \frac{b_{32}(\mu, \xi_1) c_{13}(\mu, \xi_2)}{b_{21}(\xi_1, \xi_2)} \underbrace{c_{21}(\xi_1, \xi_2)}_{\text{apply Eq.(10)-}\{2,1\}} \\
&= \frac{b_{23}(\mu, \xi_1) c_{13}(\mu, \xi_2) c_{12}(\xi_2, \xi_1)}{b_{21}(\xi_2, \xi_1)} \\
\kappa_8^{(B_1)} &= \frac{1}{b_{21}(\xi_1, \xi_2)} \underbrace{\{c_{23}(\mu, \xi_1) c_{12}(\mu, \xi_2) b_{32}(\xi_1, \xi_2) + b_{32}(\mu, \xi_1) c_{13}(\mu, \xi_2) c_{32}(\xi_1, \xi_2)\}}_{\text{apply Eq.(21)-}\{1,3,2\}} \\
&= \frac{c_{13}(\mu, \xi_1) b_{32}(\mu, \xi_2) c_{12}(\xi_1, \xi_2)}{b_{21}(\xi_1, \xi_2)}
\end{aligned}$$

Appendix C: The analysis for $L = 3$

In what follows we present the explicit expressions of the entries (93) together with the corresponding simplifications using the Yang-Baxter and unitarity relations;

$$\begin{aligned}
\alpha_1^{(C_1)} &= \frac{c_{21}(\mu, \xi_1) c_{32}(\mu, \xi_2) \overbrace{b_{31}(\mu, \xi_3) b_{21}(\xi_2, \xi_3)}^{\text{apply Eq. (13)-}\{3,2,1\}}}{b_{21}(\xi_1, \xi_3) b_{31}(\xi_2, \xi_3)} - \frac{c_{31}(\mu, \xi_1) b_{32}(\mu, \xi_2) b_{31}(\mu, \xi_3) \overbrace{c_{32}(\xi_1, \xi_2)}^{\text{apply Eq. (10)-}\{3,2\}}}{b_{31}(\xi_1, \xi_3) b_{32}(\xi_1, \xi_2)} \\
&= \frac{c_{21}(\mu, \xi_1) c_{32}(\mu, \xi_2) b_{21}(\mu, \xi_3)}{b_{21}(\xi_1, \xi_3)} + \frac{c_{31}(\mu, \xi_1) b_{32}(\mu, \xi_2) \overbrace{b_{31}(\mu, \xi_3) b_{21}(\xi_1, \xi_3)}^{\text{apply Eq. (13)-}\{3,2,1\}} c_{23}(\xi_2, \xi_1)}{b_{32}(\xi_2, \xi_1) b_{31}(\xi_1, \xi_3) b_{21}(\xi_1, \xi_3)} \\
&= \frac{b_{21}(\mu, \xi_3)}{b_{32}(\xi_2, \xi_1) b_{21}(\xi_1, \xi_3)} \underbrace{\{c_{32}(\mu, \xi_2) c_{21}(\mu, \xi_1) b_{32}(\xi_2, \xi_1) + b_{32}(\mu, \xi_2) c_{31}(\mu, \xi_1) c_{23}(\xi_2, \xi_1)\}}_{\text{apply Eq. (20)-}\{1,3,2\}} \\
&= \frac{b_{21}(\mu, \xi_3) c_{31}(\mu, \xi_2) b_{32}(\mu, \xi_1) c_{21}(\xi_2, \xi_1)}{b_{32}(\xi_2, \xi_1) b_{21}(\xi_1, \xi_3)} \\
\alpha_2^{(C_1)} &= \frac{c_{21}(\mu, \xi_1) c_{32}(\mu, \xi_2) b_{32}(\mu, \xi_3)}{b_{32}(\xi_2, \xi_3)} + \frac{c_{31}(\mu, \xi_1) b_{32}(\mu, \xi_2) b_{32}(\mu, \xi_3) \overbrace{c_{32}(\xi_1, \xi_2)}^{\text{apply Eq. (10)-}\{3,2\}}}{b_{32}(\xi_2, \xi_3) b_{32}(\xi_1, \xi_2)} \\
&= \frac{b_{32}(\mu, \xi_3)}{b_{32}(\xi_2, \xi_1) b_{32}(\xi_2, \xi_3)} \underbrace{\{c_{32}(\mu, \xi_2) c_{21}(\mu, \xi_1) b_{32}(\xi_2, \xi_1) + b_{32}(\mu, \xi_2) c_{31}(\mu, \xi_1) c_{23}(\xi_2, \xi_1)\}}_{\text{apply Eq. (20)-}\{1,3,2\}} \\
&= \frac{b_{32}(\mu, \xi_3) c_{31}(\mu, \xi_2) b_{32}(\mu, \xi_1) c_{21}(\xi_2, \xi_1)}{b_{32}(\xi_2, \xi_1) b_{32}(\xi_2, \xi_3)} \\
\alpha_3^{(C_1)} &= c_{21}(\mu, \xi_1) c_{32}(\mu, \xi_2) a_3(\mu, \xi_3) - \frac{c_{31}(\mu, \xi_1) b_{32}(\mu, \xi_2) a_3(\mu, \xi_3) \overbrace{c_{32}(\xi_1, \xi_2)}^{\text{apply Eq. (10)-}\{3,2\}}}{b_{32}(\xi_1, \xi_2)} \\
&= \frac{a_3(\mu, \xi_3)}{b_{32}(\xi_2, \xi_1)} \underbrace{\{c_{32}(\mu, \xi_2) c_{21}(\mu, \xi_1) b_{32}(\xi_2, \xi_1) + b_{32}(\mu, \xi_2) c_{31}(\mu, \xi_1) c_{23}(\xi_2, \xi_1)\}}_{\text{apply Eq. (20)-}\{1,3,2\}} \\
&= \frac{a_3(\mu, \xi_3) c_{31}(\mu, \xi_2) b_{32}(\mu, \xi_1) c_{21}(\xi_2, \xi_1)}{b_{32}(\xi_2, \xi_1)} \\
\beta_1^{(C_1)} &= \frac{c_{21}(\mu, \xi_1) b_{21}(\mu, \xi_2) c_{32}(\mu, \xi_3)}{b_{21}(\xi_1, \xi_2)} - \frac{c_{31}(\mu, \xi_1) \overbrace{b_{31}(\mu, \xi_2) b_{21}(\xi_1, \xi_2)}^{\text{apply Eq. (13)-}\{3,2,1\}} b_{32}(\mu, \xi_3) \overbrace{c_{32}(\xi_1, \xi_3)}^{\text{apply Eq. (10)-}\{3,2\}}}{b_{31}(\xi_1, \xi_2) b_{21}(\xi_1, \xi_2) b_{32}(\xi_1, \xi_3)} \\
&= \frac{b_{21}(\mu, \xi_2)}{b_{21}(\xi_1, \xi_2) b_{32}(\xi_3, \xi_1)} \underbrace{\{c_{32}(\mu, \xi_3) c_{21}(\mu, \xi_1) b_{32}(\xi_3, \xi_1) + b_{32}(\mu, \xi_3) c_{31}(\mu, \xi_1) c_{23}(\xi_3, \xi_1)\}}_{\text{apply Eq. (20)-}\{1,3,2\}} \\
&= \frac{b_{21}(\mu, \xi_2) c_{31}(\mu, \xi_3) b_{32}(\mu, \xi_1) c_{21}(\xi_3, \xi_1)}{b_{21}(\xi_1, \xi_2) b_{32}(\xi_3, \xi_1)}
\end{aligned}$$

$$\begin{aligned}
\beta_2^{(C_1)} &= c_{21}(\mu, \xi_1) a_2(\mu, \xi_2) c_{32}(\mu, \xi_3) - \frac{c_{21}(\mu, \xi_1) c_{32}(\mu, \xi_2) b_{32}(\mu, \xi_3) \overbrace{c_{32}(\xi_2, \xi_3)}^{\text{apply Eq.(10)-}\{3,2\}}}{b_{32}(\xi_2, \xi_3)} \\
&\quad + \frac{c_{31}(\mu, \xi_1) b_{32}(\mu, \xi_2) b_{32}(\mu, \xi_3) c_{32}(\xi_1, \xi_2) \overbrace{c_{32}(\xi_2, \xi_3)}^{\text{apply Eq.(10)-}\{3,2\}}}{b_{32}(\xi_1, \xi_2) b_{32}(\xi_2, \xi_3)} - \frac{c_{31}(\mu, \xi_1) b_{32}(\mu, \xi_2) b_{32}(\mu, \xi_3) a_2(\xi_1, \xi_2) c_{32}(\xi_1, \xi_3)}{b_{32}(\xi_1, \xi_2) b_{32}(\xi_1, \xi_3)} \\
&= \frac{c_{21}(\mu, \xi_1)}{b_{32}(\xi_3, \xi_2)} \underbrace{\{c_{32}(\mu, \xi_3) a_2(\mu, \xi_2) b_{32}(\xi_3, \xi_2) + b_{32}(\mu, \xi_3) c_{32}(\mu, \xi_2) c_{23}(\xi_3, \xi_2)\}}_{\text{apply Eq.(19)-}\{3,2\}} \\
&\quad - \frac{c_{31}(\mu, \xi_1) b_{32}(\mu, \xi_2) b_{32}(\mu, \xi_3)}{b_{32}(\xi_1, \xi_2) b_{32}(\xi_3, \xi_2) b_{32}(\xi_1, \xi_3)} \underbrace{\{b_{32}(\xi_1, \xi_3) c_{32}(\xi_1, \xi_2) c_{23}(\xi_3, \xi_2) + c_{32}(\xi_1, \xi_3) a_2(\xi_1, \xi_2) b_{32}(\xi_3, \xi_2)\}}_{\text{apply Eq.(19)-}\{3,2\}} \\
&= \frac{c_{21}(\mu, \xi_1) c_{32}(\mu, \xi_3) b_{32}(\mu, \xi_2) a_2(\xi_3, \xi_2)}{b_{32}(\xi_3, \xi_2)} - \frac{c_{31}(\mu, \xi_1) b_{32}(\mu, \xi_2) b_{32}(\mu, \xi_3) \overbrace{c_{32}(\xi_1, \xi_3)}^{\text{apply Eq.(10)-}\{3,2\}} a_2(\xi_3, \xi_2)}{b_{32}(\xi_1, \xi_3) b_{32}(\xi_3, \xi_2)} \\
&= \frac{b_{32}(\mu, \xi_2) a_2(\xi_3, \xi_2)}{b_{32}(\xi_3, \xi_2) b_{32}(\xi_3, \xi_1)} \underbrace{\{c_{32}(\mu, \xi_3) c_{21}(\mu, \xi_1) b_{32}(\xi_3, \xi_1) + b_{32}(\mu, \xi_3) c_{31}(\mu, \xi_1) c_{23}(\xi_3, \xi_1)\}}_{\text{apply Eq.(20)-}\{1,3,2\}} \\
&= \frac{b_{32}(\mu, \xi_2) a_2(\xi_3, \xi_2) c_{31}(\mu, \xi_3) b_{32}(\mu, \xi_1) c_{21}(\xi_3, \xi_1)}{b_{32}(\xi_3, \xi_2) b_{32}(\xi_3, \xi_1)} \\
\beta_3^{(C_1)} &= c_{21}(\mu, \xi_1) \underbrace{\{c_{32}(\mu, \xi_2) a_3(\mu, \xi_3) c_{32}(\xi_2, \xi_3) + b_{23}(\mu, \xi_2) c_{32}(\mu, \xi_3) b_{32}(\xi_2, \xi_3)\}}_{\text{apply Eq.(16)-}\{3,2\}} \\
&\quad - \frac{c_{31}(\mu, \xi_1) c_{32}(\xi_1, \xi_2)}{b_{32}(\xi_1, \xi_2)} \underbrace{\{b_{32}(\mu, \xi_2) a_3(\mu, \xi_3) c_{32}(\xi_2, \xi_3) + c_{23}(\mu, \xi_2) c_{32}(\mu, \xi_3) b_{32}(\xi_2, \xi_3)\}}_{\text{apply Eq.(14)-}\{3,2\}} \\
&\quad + \frac{c_{31}(\mu, \xi_1) a_3(\mu, \xi_2) b_{32}(\mu, \xi_3) \overbrace{c_{32}(\xi_1, \xi_2)}^{\text{apply Eq.(10)-}\{3,2\}} c_{32}(\xi_2, \xi_3)}{b_{32}(\xi_1, \xi_2)} - \frac{c_{31}(\mu, \xi_1) a_3(\mu, \xi_2) b_{32}(\mu, \xi_3) c_{32}(\xi_1, \xi_3) a_3(\xi_2, \xi_3)}{b_{32}(\xi_1, \xi_3)} \\
&= c_{21}(\mu, \xi_1) a_3(\mu, \xi_2) c_{32}(\mu, \xi_3) a_3(\xi_2, \xi_3) - \frac{c_{31}(\mu, \xi_1) c_{32}(\xi_1, \xi_2) a_3(\mu, \xi_2) b_{32}(\mu, \xi_3) c_{32}(\xi_1, \xi_3)}{b_{32}(\xi_1, \xi_2)} \\
&\quad - \frac{c_{31}(\mu, \xi_1) a_3(\mu, \xi_2) b_{32}(\mu, \xi_3)}{b_{32}(\xi_1, \xi_3) b_{32}(\xi_2, \xi_1)} \underbrace{\{c_{23}(\xi_2, \xi_1) c_{32}(\xi_2, \xi_3) b_{32}(\xi_1, \xi_3) + b_{32}(\xi_2, \xi_1) a_3(\xi_2, \xi_3) c_{32}(\xi_1, \xi_3)\}}_{\text{apply Eq.(14)-}\{3,2\}} \\
&= c_{21}(\mu, \xi_1) a_3(\mu, \xi_2) c_{32}(\mu, \xi_3) a_3(\xi_2, \xi_3) - \frac{c_{31}(\mu, \xi_1) \overbrace{c_{32}(\xi_1, \xi_2)}^{\text{apply Eq.(10)-}\{3,2\}} a_3(\mu, \xi_2) b_{32}(\mu, \xi_3) c_{32}(\xi_1, \xi_3)}{b_{32}(\xi_1, \xi_2)} \\
&\quad - \frac{c_{31}(\mu, \xi_1) a_3(\mu, \xi_2) b_{32}(\mu, \xi_3) a_3(\xi_2, \xi_1) b_{32}(\xi_2, \xi_3) \overbrace{c_{32}(\xi_1, \xi_3)}^{\text{apply Eq.(10)-}\{3,2\}}}{b_{32}(\xi_1, \xi_3) b_{32}(\xi_2, \xi_1)} \\
&= c_{21}(\mu, \xi_1) a_3(\mu, \xi_2) c_{32}(\mu, \xi_3) a_3(\xi_2, \xi_3) \\
&\quad + \frac{c_{31}(\mu, \xi_1) a_3(\mu, \xi_2) b_{32}(\mu, \xi_3)}{b_{32}(\xi_3, \xi_1) b_{32}(\xi_2, \xi_1)} \underbrace{\{c_{32}(\xi_2, \xi_3) c_{23}(\xi_2, \xi_1) b_{32}(\xi_3, \xi_1) + b_{32}(\xi_2, \xi_3) a_3(\xi_2, \xi_1) c_{23}(\xi_3, \xi_1)\}}_{\text{apply Eq.(15)-}\{3,2\}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{a_3(\mu, \xi_2)a_3(\xi_2, \xi_3)}{b_{32}(\xi_3, \xi_1)} \underbrace{\{c_{32}(\mu, \xi_3)c_{21}(\mu, \xi_1)b_{32}(\xi_3, \xi_1) + b_{32}(\mu, \xi_3)c_{31}(\mu, \xi_1)c_{23}(\xi_3, \xi_1)\}}_{\text{apply Eq.(20)-}\{1,3,2\}} \\
&= \frac{a_3(\mu, \xi_2)a_3(\xi_2, \xi_3)c_{31}(\mu, \xi_3)b_{32}(\mu, \xi_1)c_{21}(\xi_3, \xi_1)}{b_{32}(\xi_3, \xi_1)} \\
&\quad \underbrace{\quad}_{\text{apply Eq.(10)-}\{2,1\}} \\
\gamma_1^{(C_1)} &= a_1(\mu, \xi_1)c_{21}(\mu, \xi_2)c_{32}(\mu, \xi_3) - \frac{c_{21}(\mu, \xi_1)b_{21}(\mu, \xi_2)c_{32}(\mu, \xi_3)}{b_{21}(\xi_1, \xi_2)} \underbrace{c_{21}(\xi_1, \xi_2)}_{\text{apply Eq.(10)-}\{3,1\} \text{ apply Eq.(10)-}\{3,2\}} \\
&\quad + \frac{c_{31}(\mu, \xi_1)b_{31}(\mu, \xi_2)b_{32}(\mu, \xi_3)}{b_{31}(\xi_1, \xi_2)b_{32}(\xi_1, \xi_2)} \underbrace{c_{31}(\xi_1, \xi_2)}_{\text{apply Eq.(10)-}\{3,1\}} \underbrace{c_{32}(\xi_2, \xi_3)}_{\text{apply Eq.(10)-}\{3,2\}} - \frac{a_1(\mu, \xi_1)c_{31}(\mu, \xi_2)b_{32}(\mu, \xi_3)}{b_{32}(\xi_2, \xi_3)} \underbrace{c_{32}(\xi_2, \xi_3)}_{\text{apply Eq.(10)-}\{3,2\}} \\
&= \frac{c_{32}(\mu, \xi_3)}{b_{21}(\xi_2, \xi_1)} \underbrace{\{c_{21}(\mu, \xi_2)a_1(\mu, \xi_1)b_{21}(\xi_2, \xi_1) + b_{21}(\mu, \xi_2)c_{21}(\mu, \xi_1)c_{12}(\xi_2, \xi_1)\}}_{\text{apply Eq.(19)-}\{2,1\}} \\
&\quad + \frac{b_{32}(\mu, \xi_3)c_{23}(\xi_3, \xi_2)}{b_{31}(\xi_2, \xi_1)b_{32}(\xi_3, \xi_2)} \underbrace{\{b_{31}(\mu, \xi_2)c_{31}(\mu, \xi_1)c_{31}(\xi_2, \xi_1) + c_{31}(\mu, \xi_2)a_1(\mu, \xi_1)b_{31}(\xi_2, \xi_1)\}}_{\text{apply Eq.(19)-}\{3,1\}} \\
&\quad \underbrace{\quad}_{\text{apply Eq.(13)-}\{3,2,1\}} \\
&= \frac{c_{32}(\mu, \xi_3)c_{21}(\mu, \xi_2)b_{21}(\mu, \xi_1)a_1(\xi_2, \xi_1)}{b_{21}(\xi_2, \xi_1)} + \frac{b_{32}(\mu, \xi_3)c_{23}(\xi_3, \xi_2)c_{31}(\mu, \xi_2)a_1(\xi_2, \xi_1)}{b_{32}(\xi_3, \xi_2)b_{31}(\xi_2, \xi_1)} \underbrace{b_{31}(\mu, \xi_1)}_{\text{apply Eq.(13)-}\{3,2,1\}} \\
&= \frac{b_{21}(\mu, \xi_1)a_1(\xi_2, \xi_1)}{b_{21}(\xi_2, \xi_1)b_{32}(\xi_3, \xi_2)} \underbrace{\{c_{32}(\mu, \xi_3)c_{21}(\mu, \xi_2)b_{32}(\xi_3, \xi_2) + b_{32}(\mu, \xi_3)c_{31}(\mu, \xi_2)c_{23}(\xi_3, \xi_2)\}}_{\text{apply Eq.(20)-}\{1,3,2\}} \\
&= \frac{b_{21}(\mu, \xi_1)a_1(\xi_2, \xi_1)c_{31}(\mu, \xi_3)b_{32}(\mu, \xi_2)c_{21}(\xi_3, \xi_2)}{b_{21}(\xi_2, \xi_1)b_{32}(\xi_3, \xi_2)} \\
\gamma_2^{(C_1)} &= b_{12}(\mu, \xi_1)c_{21}(\mu, \xi_2)c_{32}(\mu, \xi_3)c_{21}(\xi_1, \xi_2) - \frac{b_{12}(\mu, \xi_1)c_{31}(\mu, \xi_2)b_{32}(\mu, \xi_3)b_{21}(\xi_1, \xi_2)}{b_{32}(\xi_1, \xi_2)} \underbrace{c_{32}(\xi_1, \xi_2)}_{\text{apply Eq.(10)-}\{3,2\}} \\
&\quad + c_{21}(\mu, \xi_1)a_2(\mu, \xi_2)c_{32}(\mu, \xi_3)c_{21}(\xi_1, \xi_2) - \frac{c_{21}(\mu, \xi_1)c_{32}(\mu, \xi_2)b_{32}(\mu, \xi_3)c_{21}(\xi_1, \xi_2)}{b_{32}(\xi_2, \xi_3)} \underbrace{c_{32}(\xi_2, \xi_3)}_{\text{apply Eq.(10)-}\{3,2\}} \\
&\quad \underbrace{\quad}_{\text{apply Eq.(10)-}\{3,2\}} \\
&\quad + \frac{c_{31}(\mu, \xi_1)b_{32}(\mu, \xi_2)b_{32}(\mu, \xi_3)c_{32}(\xi_1, \xi_2)c_{21}(\xi_1, \xi_2)}{b_{32}(\xi_1, \xi_2)b_{32}(\xi_2, \xi_3)} \underbrace{c_{32}(\xi_2, \xi_3)}_{\text{apply Eq.(10)-}\{3,2\}} - \frac{c_{31}(\mu, \xi_1)b_{32}(\mu, \xi_2)b_{32}(\mu, \xi_3)a_2(\xi_1, \xi_2)c_{21}(\xi_1, \xi_2)c_{32}(\xi_1, \xi_3)}{b_{32}(\xi_1, \xi_2)b_{32}(\xi_1, \xi_3)} \\
&= \frac{b_{12}(\mu, \xi_1)b_{21}(\xi_1, \xi_2)}{b_{32}(\xi_3, \xi_2)} \underbrace{\{c_{32}(\mu, \xi_3)c_{21}(\mu, \xi_2)b_{32}(\xi_3, \xi_2) + b_{32}(\mu, \xi_3)c_{31}(\mu, \xi_2)c_{23}(\xi_3, \xi_2)\}}_{\text{apply Eq.(20)-}\{1,3,2\}} \\
&\quad \underbrace{\quad}_{\text{apply Eq.(20)-}\{1,3,2\}} \\
&\quad + \frac{c_{21}(\mu, \xi_1)c_{21}(\xi_1, \xi_2)}{b_{32}(\xi_3, \xi_2)} \underbrace{\{c_{32}(\mu, \xi_3)a_2(\mu, \xi_2)b_{32}(\xi_3, \xi_2) + b_{32}(\mu, \xi_3)c_{32}(\mu, \xi_2)c_{23}(\xi_3, \xi_2)\}}_{\text{apply Eq.(19)-}\{3,2\}} \\
&\quad \underbrace{\quad}_{\text{apply Eq.(19)-}\{3,2\}} \\
&\quad - \frac{c_{31}(\mu, \xi_1)b_{32}(\mu, \xi_2)b_{32}(\mu, \xi_3)c_{21}(\xi_1, \xi_2)}{b_{32}(\xi_1, \xi_2)b_{32}(\xi_1, \xi_2)b_{32}(\xi_3, \xi_2)} \underbrace{\{b_{32}(\xi_1, \xi_3)c_{32}(\xi_1, \xi_2)c_{23}(\xi_3, \xi_2) + c_{32}(\xi_1, \xi_3)a_2(\xi_1, \xi_2)b_{32}(\xi_3, \xi_2)\}}_{\text{apply Eq.(19)-}\{3,2\}} \\
&\quad \underbrace{\quad}_{\text{apply Eq.(19)-}\{3,2\}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{b_{12}(\mu, \xi_1)b_{21}(\xi_1, \xi_2)c_{31}(\mu, \xi_3)b_{32}(\mu, \xi_2)c_{21}(\xi_3, \xi_2)}{b_{21}(\xi_3, \xi_2)} + \frac{c_{21}(\mu, \xi_1)c_{21}(\xi_1, \xi_2)c_{32}(\mu, \xi_3)b_{32}(\mu, \xi_2)a_2(\xi_3, \xi_2)}{b_{32}(\xi_3, \xi_2)} \\
&\quad \text{apply Eq.(10)-}\{3,2\} \\
&= \frac{c_{31}(\mu, \xi_1)b_{32}(\mu, \xi_2)b_{32}(\mu, \xi_3)c_{21}(\xi_1, \xi_2)}{b_{32}(\xi_1, \xi_3)b_{32}(\xi_3, \xi_2)} \overbrace{C_{32}(\xi_1, \xi_3)}^{a_2(\xi_3, \xi_2)} \\
&= \frac{b_{12}(\mu, \xi_1)b_{21}(\xi_1, \xi_2)c_{31}(\mu, \xi_3)b_{32}(\mu, \xi_2)c_{21}(\xi_3, \xi_2)}{b_{21}(\xi_3, \xi_2)} \\
&\quad + \frac{c_{21}(\xi_1, \xi_2)a_2(\xi_3, \xi_2)b_{32}(\mu, \xi_2)}{b_{32}(\xi_3, \xi_2)b_{32}(\xi_3, \xi_1)} \underbrace{\{c_{32}(\mu, \xi_3)c_{21}(\mu, \xi_1)b_{32}(\xi_3, \xi_1) + b_{32}(\mu, \xi_3)c_{31}(\mu, \xi_1)c_{23}(\xi_3, \xi_1)\}}_{\text{apply Eq.(20)-}\{1,3,2\}} \\
&\quad \text{apply Eq.(13)-}\{1,3,2\} \\
&= \frac{\overbrace{b_{12}(\mu, \xi_1)b_{32}(\xi_3, \xi_1)}^{b_{21}(\xi_1, \xi_2)c_{31}(\mu, \xi_3)b_{32}(\mu, \xi_2)c_{21}(\xi_3, \xi_2)}}{b_{32}(\xi_3, \xi_1)b_{32}(\xi_3, \xi_2)} + \frac{c_{31}(\mu, \xi_3)b_{32}(\mu, \xi_2)c_{21}(\xi_1, \xi_2)a_2(\xi_3, \xi_2)b_{32}(\mu, \xi_1)c_{21}(\xi_3, \xi_1)}{b_{32}(\xi_3, \xi_2)b_{32}(\xi_3, \xi_1)} \\
&= \frac{c_{31}(\mu, \xi_3)b_{32}(\mu, \xi_2)b_{32}(\mu, \xi_1)}{b_{32}(\xi_3, \xi_2)b_{32}(\xi_3, \xi_1)} \underbrace{\{b_{12}(\xi_3, \xi_1)c_{21}(\xi_3, \xi_2)b_{21}(\xi_1, \xi_2) + c_{21}(\xi_3, \xi_1)a_2(\xi_3, \xi_2)c_{21}(\xi_1, \xi_2)\}}_{\text{apply Eq.(16)-}\{2,1\}} \\
&= \frac{c_{31}(\mu, \xi_3)b_{32}(\mu, \xi_2)b_{32}(\mu, \xi_1)a_2(\xi_3, \xi_1)a_2(\xi_1, \xi_2)c_{21}(\xi_3, \xi_2)}{b_{21}(\xi_3, \xi_1)b_{32}(\xi_3, \xi_2)} \\
\gamma_3^{(C_1)} &= \frac{b_{13}(\mu, \xi_1)c_{21}(\mu, \xi_2)c_{32}(\mu, \xi_3)b_{31}(\xi_1, \xi_2)b_{32}(\xi_1, \xi_2)}{b_{32}(\xi_1, \xi_2)} - \frac{b_{13}(\mu, \xi_1)c_{31}(\mu, \xi_2)b_{32}(\mu, \xi_3)b_{31}(\xi_1, \xi_2)b_{32}(\xi_1, \xi_3)}{b_{32}(\xi_1, \xi_2)b_{32}(\xi_2, \xi_3)} \overbrace{C_{32}(\xi_2, \xi_3)}^{\text{apply Eq.(10)-}\{3,2\}} \\
&\quad \text{apply Eq.(10)-}\{3,2\} \\
&\quad + \frac{c_{31}(\mu, \xi_1)c_{23}(\mu, \xi_2)c_{32}(\mu, \xi_3)c_{31}(\xi_1, \xi_2)b_{32}(\xi_1, \xi_3)}{b_{32}(\xi_1, \xi_2)} - \frac{c_{31}(\mu, \xi_1)a_3(\mu, \xi_2)b_{32}(\mu, \xi_3)c_{31}(\xi_1, \xi_2)b_{32}(\xi_1, \xi_3)}{b_{32}(\xi_1, \xi_2)b_{32}(\xi_2, \xi_3)} \overbrace{C_{32}(\xi_2, \xi_3)}^{\text{apply Eq.(10)-}\{3,2\}} \\
&\quad + \frac{c_{31}(\mu, \xi_1)b_{32}(\mu, \xi_2)a_3(\mu, \xi_3)c_{21}(\xi_1, \xi_2)c_{32}(\xi_1, \xi_3)}{b_{32}(\xi_1, \xi_2)} \\
&= \frac{b_{13}(\mu, \xi_1)b_{31}(\xi_1, \xi_2)b_{32}(\xi_1, \xi_3)}{b_{32}(\xi_1, \xi_2)b_{12}(\xi_3, \xi_2)} \underbrace{\{c_{32}(\mu, \xi_3)c_{21}(\mu, \xi_2)b_{32}(\xi_3, \xi_2) + b_{32}(\mu, \xi_3)c_{31}(\mu, \xi_2)c_{23}(\xi_3, \xi_2)\}}_{\text{apply Eq.(20)-}\{1,3,2\}} \\
&\quad + \frac{c_{31}(\mu, \xi_1)c_{31}(\xi_1, \xi_2)b_{32}(\xi_1, \xi_3)}{b_{32}(\xi_1, \xi_2)b_{32}(\xi_3, \xi_2)} \underbrace{\{c_{32}(\mu, \xi_3)c_{23}(\mu, \xi_2)b_{32}(\xi_3, \xi_2) + b_{32}(\mu, \xi_3)a_3(\mu, \xi_2)c_{23}(\xi_3, \xi_2)\}}_{\text{apply Eq.(15)-}\{3,2\}} \\
&\quad + \frac{c_{31}(\mu, \xi_1)b_{32}(\mu, \xi_2)a_3(\mu, \xi_3)c_{21}(\xi_1, \xi_2)c_{32}(\xi_1, \xi_3)}{b_{32}(\xi_1, \xi_2)} \\
&= \frac{b_{13}(\mu, \xi_1)b_{31}(\xi_1, \xi_2)b_{32}(\xi_1, \xi_3)c_{31}(\mu, \xi_3)b_{32}(\mu, \xi_2)c_{21}(\xi_3, \xi_2)}{b_{32}(\xi_1, \xi_2)b_{32}(\xi_3, \xi_2)} \\
&\quad + \frac{c_{31}(\mu, \xi_1)b_{32}(\mu, \xi_2)a_3(\mu, \xi_3)}{b_{32}(\xi_1, \xi_2)b_{32}(\xi_3, \xi_2)} \underbrace{\{b_{32}(\xi_1, \xi_3)c_{31}(\xi_1, \xi_2)c_{23}(\xi_3, \xi_2) + c_{32}(\xi_1, \xi_3)c_{21}(\xi_1, \xi_2)b_{32}(\xi_3, \xi_2)\}}_{\text{apply Eq.(20)-}\{1,3,2\}} \\
&\quad \text{apply Eq.(12)-}\{3,2,1\} \\
&= \frac{b_{13}(\mu, \xi_1)b_{32}(\mu, \xi_2)c_{31}(\mu, \xi_3)}{b_{32}(\xi_3, \xi_2)b_{32}(\xi_1, \xi_2)} \overbrace{b_{31}(\xi_1, \xi_2)}^{b_{32}(\xi_1, \xi_3)c_{21}(\xi_3, \xi_2)} + \frac{c_{31}(\mu, \xi_1)b_{32}(\mu, \xi_2)a_3(\mu, \xi_3)c_{31}(\xi_1, \xi_3)c_{21}(\xi_3, \xi_2)}{b_{32}(\xi_3, \xi_2)} \\
&= \frac{b_{32}(\mu, \xi_2)c_{21}(\xi_3, \xi_2)}{b_{32}(\xi_3, \xi_2)} \underbrace{\{b_{13}(\mu, \xi_1)c_{31}(\mu, \xi_3)b_{31}(\xi_1, \xi_3) + c_{31}(\mu, \xi_1)a_3(\mu, \xi_3)c_{31}(\xi_1, \xi_3)\}}_{\text{apply Eq.(16)-}\{3,1\}} \\
&= \frac{c_{32}(\mu, \xi_2)c_{21}(\xi_3, \xi_2)a_3(\mu, \xi_1)c_{31}(\mu, \xi_3)a_3(\xi_1, \xi_3)}{b_{32}(\xi_3, \xi_2)}
\end{aligned}$$

$$\begin{aligned}
\delta_1^{(C_1)} &= \frac{\overbrace{c_{31}(\mu, \xi_1) b_{32}(\mu, \xi_2) b_{31}(\mu, \xi_3) b_{21}(\xi_1, \xi_3)}^{\text{apply Eq. (13)-}\{3,2,1\}} c_{21}(\xi_1, \xi_2)}{b_{32}(\xi_1, \xi_2) b_{31}(\xi_1, \xi_3) b_{21}(\xi_2, \xi_3)} \\
&= \frac{c_{31}(\mu, \xi_1) b_{32}(\mu, \xi_2) b_{21}(\mu, \xi_3) c_{21}(\xi_1, \xi_2)}{b_{32}(\xi_1, \xi_2) b_{21}(\xi_2, \xi_3)} \\
\delta_2^{(C_1)} &= \frac{c_{31}(\mu, \xi_1) b_{32}(\mu, \xi_2) b_{32}(\mu, \xi_3) c_{21}(\xi_1, \xi_2)}{b_{32}(\xi_1, \xi_2) b_{32}(\xi_1, \xi_3)} \\
\delta_3^{(C_1)} &= \frac{c_{31}(\mu, \xi_1) b_{32}(\mu, \xi_2) a_3(\mu, \xi_3) c_{21}(\xi_1, \xi_2)}{b_{32}(\xi_1, \xi_2)} \\
\phi_1^{(C_1)} &= \frac{c_{31}(\mu, \xi_1) b_{31}(\mu, \xi_2) b_{32}(\mu, \xi_3) a_1(\xi_1, \xi_2) c_{21}(\xi_1, \xi_3)}{b_{31}(\xi_1, \xi_2) b_{32}(\xi_1, \xi_3)} - \frac{\overbrace{c_{31}(\mu, \xi_1) b_{32}(\mu, \xi_2) b_{31}(\mu, \xi_3) c_{21}(\xi_1, \xi_2) b_{21}(\xi_1, \xi_3)}^{\text{apply Eq. (12)-}\{3,2,1\}} \overbrace{c_{21}(\xi_2, \xi_3)}^{\text{apply Eq. (10)-}\{2,1\}}}{b_{31}(\xi_1, \xi_2) b_{32}(\xi_1, \xi_3) b_{21}(\xi_2, \xi_3)} \\
&= \frac{c_{31}(\mu, \xi_1) b_{31}(\mu, \xi_2) b_{32}(\mu, \xi_3)}{b_{31}(\xi_1, \xi_2) b_{32}(\xi_1, \xi_3) b_{21}(\xi_3, \xi_2)} \underbrace{\{c_{21}(\xi_1, \xi_3) a_1(\xi_1, \xi_2) b_{21}(\xi_3, \xi_2) + b_{21}(\xi_1, \xi_3) c_{21}(\xi_1, \xi_2) c_{12}(\xi_3, \xi_2)\}}_{\text{apply Eq. (19)-}\{2,1\}} \\
&= \frac{\overbrace{c_{31}(\mu, \xi_1) b_{31}(\mu, \xi_2) b_{21}(\xi_1, \xi_2) b_{32}(\mu, \xi_3) c_{21}(\xi_1, \xi_3) a_1(\xi_3, \xi_2)}^{\text{apply Eq. (13)-}\{3,2,1\}}}{b_{31}(\xi_1, \xi_2) b_{32}(\xi_1, \xi_3) b_{21}(\xi_3, \xi_2)} \\
&= \frac{c_{31}(\mu, \xi_1) b_{21}(\mu, \xi_2) b_{32}(\mu, \xi_3) c_{21}(\xi_1, \xi_3) a_1(\xi_3, \xi_2)}{b_{32}(\xi_1, \xi_3) b_{21}(\xi_3, \xi_2)} \\
\phi_2^{(C_1)} &= \frac{c_{31}(\mu, \xi_1) b_{32}(\mu, \xi_2) b_{32}(\mu, \xi_3) c_{21}(\xi_1, \xi_3) a_2(\xi_2, \xi_3)}{b_{32}(\xi_1, \xi_2) b_{32}(\xi_1, \xi_3)} \\
\phi_3^{(C_1)} &= \frac{c_{31}(\mu, \xi_1) a_3(\mu, \xi_2) b_{32}(\mu, \xi_3) c_{21}(\xi_1, \xi_3)}{b_{32}(\xi_1, \xi_3)} \\
\omega_1^{(C_1)} &= \frac{a_1(\mu, \xi_1) c_{31}(\mu, \xi_2) b_{32}(\mu, \xi_3) c_{21}(\xi_2, \xi_3)}{b_{32}(\xi_2, \xi_3)} - \frac{\overbrace{c_{31}(\mu, \xi_1) b_{31}(\mu, \xi_2) b_{32}(\mu, \xi_3) c_{31}(\xi_1, \xi_2)}^{\text{apply Eq. (10)-}\{3,1\}} c_{21}(\xi_2, \xi_3)}{b_{31}(\xi_1, \xi_2) b_{32}(\xi_2, \xi_3)} \\
&\quad + \frac{\overbrace{c_{31}(\mu, \xi_1) b_{32}(\mu, \xi_2) b_{31}(\mu, \xi_3) c_{31}(\xi_1, \xi_2)}^{\text{apply Eq. (10)-}\{3,1\}} c_{21}(\xi_2, \xi_3)}{b_{31}(\xi_1, \xi_2) b_{32}(\xi_2, \xi_3)} - \frac{c_{31}(\mu, \xi_1) b_{32}(\mu, \xi_2) b_{31}(\mu, \xi_3) c_{31}(\xi_1, \xi_3) c_{23}(\xi_2, \xi_3)}{b_{31}(\xi_1, \xi_3) b_{32}(\xi_2, \xi_3)} \\
&\quad - \frac{c_{21}(\mu, \xi_1) c_{32}(\mu, \xi_2) b_{31}(\mu, \xi_3) c_{21}(\xi_1, \xi_3) b_{21}(\xi_2, \xi_3)}{b_{21}(\xi_1, \xi_3) b_{31}(\xi_2, \xi_3)} \\
&= \frac{b_{32}(\mu, \xi_3) c_{21}(\xi_2, \xi_3)}{b_{32}(\xi_2, \xi_3) b_{31}(\xi_2, \xi_1)} \underbrace{\{c_{31}(\mu, \xi_2) a_1(\mu, \xi_1) b_{31}(\xi_2, \xi_1) + b_{31}(\mu, \xi_2) c_{31}(\mu, \xi_1) c_{13}(\xi_2, \xi_1)\}}_{\text{apply Eq. (19)-}\{3,1\}} \\
&\quad - \frac{c_{31}(\mu, \xi_1) b_{32}(\mu, \xi_2) b_{31}(\mu, \xi_3)}{b_{31}(\xi_2, \xi_1) b_{32}(\xi_2, \xi_3) b_{31}(\xi_1, \xi_3)} \underbrace{\{c_{13}(\xi_2, \xi_1) c_{21}(\xi_2, \xi_3) b_{31}(\xi_1, \xi_3) + b_{31}(\xi_2, \xi_1) c_{23}(\xi_2, \xi_3) c_{31}(\xi_1, \xi_3)\}}_{\text{apply Eq. (21)-}\{2,3,1\}} \\
&\quad - \frac{c_{21}(\mu, \xi_1) c_{32}(\mu, \xi_2) b_{31}(\mu, \xi_3) c_{21}(\xi_1, \xi_3) b_{21}(\xi_2, \xi_3)}{b_{21}(\xi_1, \xi_3) b_{31}(\xi_2, \xi_3)} \\
&= \frac{b_{32}(\mu, \xi_3) c_{21}(\xi_2, \xi_3) c_{31}(\mu, \xi_2) b_{31}(\mu, \xi_1) a_1(\xi_2, \xi_1)}{b_{32}(\xi_2, \xi_3) b_{31}(\xi_2, \xi_1)} - \frac{c_{31}(\mu, \xi_1) b_{32}(\mu, \xi_2) b_{31}(\mu, \xi_3) c_{32}(\xi_2, \xi_1) b_{31}(\xi_2, \xi_3) c_{21}(\xi_1, \xi_3)}{\underbrace{b_{31}(\xi_2, \xi_1) b_{32}(\xi_2, \xi_3) b_{31}(\xi_1, \xi_3)}_{\text{apply Eq. (12)-}\{3,1,2\}}} \\
&\quad - \frac{c_{21}(\mu, \xi_1) c_{32}(\mu, \xi_2) b_{31}(\mu, \xi_3) c_{21}(\xi_1, \xi_3) b_{21}(\xi_2, \xi_3)}{\underbrace{b_{21}(\xi_1, \xi_3) b_{31}(\xi_2, \xi_3)}_{\text{apply Eq. (13)-}\{2,3,1\}}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{b_{32}(\mu, \xi_3)c_{21}(\xi_2, \xi_3)c_{31}(\mu, \xi_2)b_{31}(\mu, \xi_1)a_1(\xi_2, \xi_1)}{b_{32}(\xi_2, \xi_3)b_{31}(\xi_2, \xi_1)} \\
&\quad - \frac{c_{21}(\xi_1, \xi_3)b_{31}(\mu, \xi_3)}{b_{31}(\xi_1, \xi_3)b_{32}(\xi_2, \xi_1)} \underbrace{\{b_{32}(\mu, \xi_2)c_{31}(\mu, \xi_1)c_{23}(\xi_2, \xi_1) + c_{32}(\mu, \xi_2)c_{21}(\mu, \xi_1)b_{32}(\xi_2, \xi_1)\}}_{\substack{\text{apply Eq. (20)-}\{1,3,2\} \\ \text{apply Eq. (12)-}\{3,1,2\}}} \\
&= \frac{b_{32}(\mu, \xi_3)c_{21}(\xi_2, \xi_3)c_{31}(\mu, \xi_2)b_{31}(\mu, \xi_1)a_1(\xi_2, \xi_1)}{b_{32}(\xi_2, \xi_3)b_{31}(\xi_2, \xi_1)} - \frac{c_{21}(\xi_1, \xi_3) \overbrace{b_{31}(\mu, \xi_3)b_{32}(\mu, \xi_1)}^{\text{apply Eq. (12)-}\{3,1,2\}}} c_{31}(\mu, \xi_2)c_{21}(\xi_2, \xi_1)}{b_{31}(\xi_1, \xi_3)b_{32}(\xi_2, \xi_1)} \\
&= \frac{b_{32}(\mu, \xi_3)c_{21}(\xi_2, \xi_3)c_{31}(\mu, \xi_2)b_{31}(\mu, \xi_1)a_1(\xi_2, \xi_1)}{b_{32}(\xi_2, \xi_3)b_{31}(\xi_2, \xi_1)} - \frac{b_{32}(\mu, \xi_3)c_{31}(\mu, \xi_2)c_{21}(\xi_1, \xi_3)b_{31}(\mu, \xi_1)c_{21}(\xi_2, \xi_1) \overbrace{b_{32}(\xi_2, \xi_3)b_{31}(\xi_2, \xi_1)}^{\text{apply Eq. (12)-}\{3,1,2\}}}{b_{32}(\xi_2, \xi_3)b_{31}(\xi_1, \xi_3)b_{32}(\xi_2, \xi_1)b_{31}(\xi_2, \xi_1)} \\
&= \frac{b_{32}(\mu, \xi_3)c_{21}(\xi_2, \xi_3)c_{31}(\mu, \xi_2)b_{31}(\mu, \xi_1)a_1(\xi_2, \xi_1)}{b_{32}(\xi_2, \xi_3)b_{31}(\xi_2, \xi_1)} - \frac{b_{32}(\mu, \xi_3)c_{31}(\mu, \xi_2)b_{31}(\mu, \xi_1)c_{21}(\xi_1, \xi_3)c_{21}(\xi_2, \xi_1) \overbrace{b_{31}(\xi_2, \xi_3)b_{21}(\xi_1, \xi_3)}^{\text{apply Eq. (13)-}\{2,3,1\}}}{b_{32}(\xi_2, \xi_3)b_{31}(\xi_1, \xi_3)b_{32}(\xi_2, \xi_1)b_{21}(\xi_1, \xi_3)} \\
&= \frac{b_{32}(\mu, \xi_3)c_{21}(\xi_2, \xi_3)c_{31}(\mu, \xi_2)b_{31}(\mu, \xi_1)a_1(\xi_2, \xi_1)}{b_{32}(\xi_2, \xi_3)b_{31}(\xi_2, \xi_1)} - \frac{b_{32}(\mu, \xi_3)c_{31}(\mu, \xi_2)b_{31}(\mu, \xi_1) \overbrace{c_{21}(\xi_1, \xi_3)}^{\text{apply Eq. (10)-}\{2,1\}}} c_{21}(\xi_2, \xi_1)b_{21}(\xi_2, \xi_3)}{b_{32}(\xi_2, \xi_3)b_{31}(\xi_1, \xi_3)b_{21}(\xi_1, \xi_3)} \\
&= \frac{b_{32}(\mu, \xi_3)c_{31}(\mu, \xi_2)b_{31}(\mu, \xi_1)}{b_{21}(\xi_3, \xi_1)b_{32}(\xi_2, \xi_3)b_{31}(\xi_2, \xi_1)} \underbrace{\{c_{21}(\xi_2, \xi_3)a_1(\xi_2, \xi_1)b_{21}(\xi_3, \xi_1) + b_{21}(\xi_2, \xi_3)c_{21}(\xi_2, \xi_1)c_{12}(\xi_3, \xi_1)\}}_{\text{apply Eq. (19)-}\{2,1\}} \\
&= \frac{b_{32}(\mu, \xi_3)c_{31}(\mu, \xi_2) \overbrace{b_{31}(\mu, \xi_1)b_{21}(\xi_2, \xi_1)}^{\text{apply Eq. (13)-}\{3,2,1\}}} c_{21}(\xi_2, \xi_3)a_1(\xi_3, \xi_1)}{b_{21}(\xi_3, \xi_1)b_{32}(\xi_2, \xi_3)b_{31}(\xi_2, \xi_1)} \\
&= \frac{b_{32}(\mu, \xi_3)c_{31}(\mu, \xi_2)b_{21}(\mu, \xi_1)c_{21}(\xi_2, \xi_3)a_1(\xi_3, \xi_1)}{b_{21}(\xi_3, \xi_1)b_{32}(\xi_2, \xi_3)} \\
&= \frac{c_{21}(\mu, \xi_1)c_{32}(\mu, \xi_2)b_{32}(\mu, \xi_3)c_{21}(\xi_1, \xi_3)a_2(\xi_2, \xi_3)}{b_{32}(\xi_2, \xi_3)} - \frac{c_{31}(\mu, \xi_1)b_{32}(\mu, \xi_2)b_{32}(\mu, \xi_3) \overbrace{c_{32}(\xi_1, \xi_2)}^{\text{apply Eq. (10)-}\{3,2\}}} c_{21}(\xi_1, \xi_3)a_2(\xi_2, \xi_3)}{b_{32}(\xi_1, \xi_2)b_{32}(\xi_2, \xi_3)} \\
&\quad + \frac{b_{12}(\mu, \xi_1)c_{31}(\mu, \xi_2)b_{32}(\mu, \xi_3)b_{21}(\xi_1, \xi_3)c_{21}(\xi_2, \xi_3)}{b_{32}(\xi_2, \xi_3)} \\
&= \frac{b_{32}(\mu, \xi_3)c_{21}(\xi_1, \xi_3)a_2(\xi_2, \xi_3)}{b_{32}(\xi_2, \xi_1)b_{32}(\xi_2, \xi_3)} \underbrace{\{c_{32}(\mu, \xi_2)c_{21}(\mu, \xi_1)b_{32}(\xi_2, \xi_1) + b_{32}(\mu, \xi_2)c_{31}(\mu, \xi_1)c_{23}(\xi_2, \xi_1)\}}_{\text{apply Eq. (20)-}\{1,3,2\}} \\
&\quad + \frac{b_{12}(\mu, \xi_1)c_{31}(\mu, \xi_2)b_{32}(\mu, \xi_3)b_{21}(\xi_1, \xi_3)c_{21}(\xi_2, \xi_3)}{b_{32}(\xi_2, \xi_3)} \\
&\quad \text{apply Eq. (13)-}\{3,1,2\} \\
&= \frac{\overbrace{b_{32}(\mu, \xi_1)}^{\text{apply Eq. (13)-}\{3,1,2\}}} b_{32}(\xi_2, \xi_1)}{b_{32}(\xi_2, \xi_1)} \frac{b_{32}(\mu, \xi_3)c_{21}(\xi_1, \xi_3)a_2(\xi_2, \xi_3)c_{31}(\mu, \xi_2)c_{21}(\xi_2, \xi_1)}{b_{32}(\xi_2, \xi_3)} + \frac{b_{12}(\mu, \xi_1)c_{31}(\mu, \xi_2)b_{32}(\mu, \xi_3)b_{21}(\xi_1, \xi_3)c_{21}(\xi_2, \xi_3)}{b_{32}(\xi_2, \xi_3)} \\
&= \frac{\overbrace{b_{12}(\mu, \xi_1)}^{\text{apply Eq. (13)-}\{3,1,2\}}} b_{12}(\xi_2, \xi_1)}{b_{12}(\xi_2, \xi_1)} \frac{c_{31}(\mu, \xi_2)b_{32}(\mu, \xi_3)}{b_{32}(\xi_2, \xi_3)} \underbrace{\{c_{21}(\xi_2, \xi_1)a_2(\xi_2, \xi_3)c_{21}(\xi_1, \xi_3) + b_{12}(\xi_2, \xi_1)c_{21}(\xi_2, \xi_3)b_{21}(\xi_1, \xi_3)\}}_{\text{apply Y-B (13,5)}} \\
&= \frac{c_{31}(\mu, \xi_1)b_{32}(\mu, \xi_3)b_{32}(\mu, \xi_1)a_2(\xi_2, \xi_3)c_{21}(\xi_2, \xi_3)a_2(\xi_1, \xi_3)}{b_{32}(\xi_2, \xi_1)b_{32}(\xi_2, \xi_3)}
\end{aligned}$$

$$\begin{aligned}
\omega_3^{(C_1)} &= \frac{c_{31}(\mu, \xi_1) a_3(\mu, \xi_2) b_{32}(\mu, \xi_3)}{b_{32}(\xi_1, \xi_3) b_{32}(\xi_2, \xi_3)} \underbrace{\{c_{32}(\xi_1, \xi_2) c_{21}(\xi_1, \xi_3) b_{32}(\xi_2, \xi_3) + b_{32}(\xi_1, \xi_2) c_{31}(\xi_1, \xi_3) c_{23}(\xi_2, \xi_3)\}}_{\text{apply Eq.(20)-}\{1,3,2\}} \\
&\quad + \frac{b_{13}(\mu, \xi_1) c_{31}(\mu, \xi_2) b_{32}(\mu, \xi_3)}{b_{32}(\xi_1, \xi_3) b_{32}(\xi_2, \xi_3)} \underbrace{b_{32}(\xi_1, \xi_2) b_{31}(\xi_1, \xi_3) c_{21}(\xi_2, \xi_3)}_{\text{apply Eq.(12)-}\{3,1,2\}} \\
&= \frac{b_{32}(\mu, \xi_3) c_{21}(\xi_2, \xi_3) b_{32}(\xi_1, \xi_3)}{b_{32}(\xi_1, \xi_3) b_{32}(\xi_2, \xi_3)} \underbrace{\{c_{31}(\mu, \xi_1) a_3(\mu, \xi_2) c_{31}(\xi_1, \xi_2) + b_{13}(\mu, \xi_1) c_{31}(\mu, \xi_2) b_{31}(\xi_1, \xi_2)\}}_{\text{apply Eq.(16)-}\{3,1\}} \\
&= \frac{b_{32}(\mu, \xi_3) c_{21}(\xi_2, \xi_3) a_3(\mu, \xi_1) c_{31}(\mu, \xi_2) a_3(\xi_1, \xi_2)}{b_{32}(\xi_2, \xi_3)}
\end{aligned}$$

References

- [1] R.J. Baxter, *Exactly solved models in statistical mechanics*, Academic Press, New York (1982)
- [2] L.A Takhtadzhan and L.D. Faddeev, *Russ. Math Surveys* 34 (1979) 11
- [3] V.E. Korepin, N.M. Bogoliubov and A.G. Izergin, *Quantum inverse scattering method and correlation functions*, Cambridge University Press, Cambridge (1993)
- [4] J.M. Maillet and J.Sanchez de Santos, *Amer. Math. Soc. Transl.* 201 (2000) 137
- [5] V.G Drinfeld, *Sov. Math. Dokl.* 28 (1983) 667
- [6] N. Kitanine, J.M. Maillet and V. Terras, *Nucl.Phys.B* 554 (1999) 647
- [7] V.E Korepin, *Commun. Math. Phys.* 86 (1982) 391; *Commun. Math. Phys.* 113 (1987) 177
- [8] A.G Izergin and V.E. Korepin, *Commun. Math. Phys.* 94 (1984) 67; *Commun. Math. Phys.* 99 (1985) 271
- [9] A.G. Izergin, *Sov. Phys. Dokl.*, Vol. 32 (1987) 878; A.G. Izergin, D.A. Coker and V.E Korepin, *J. Phys. A: Math. Gen.* 25 (1992) 4315
- [10] N. Kitanine, J.M. Maillet, and V. Terras, *Nucl.Phys.B* 567 (20002) 554, *J. Phys. A: Math. Gen.* 38 (2005) 7441

- [11] N. Kitanine, J.M. Maillet, N.A. Slavnov and V. Terras, *Nucl.Phys.B* 641 (2002) 487, *J. Phys. A: Math. Gen.* 38 (2005) 7441
- [12] F. Göhmann, A. Klümper and A. Seel, *J. Phys. A* 37 (2004) 7625;
- [13] V. Terras, *Lett. Math. Phys.* 48 (1999) 263
- [14] T.D Albert, H. Boos, R. Flume, R.H Poghossian and K. Ruhlig, *Lett. Math. Phys.* 53 (2000) 201
- [15] T.D Albert, H. Boos, R. Flume and K. Ruhlig, *J.Phys.A* 33 (2000) 4963
- [16] W.L. Yang, Y.Z. Zhang and S.Y. Zhao, *JHEP* 12 (2004) 038; *Commun. Math. Phys.* 264 (2006) 87
- [17] S.Y. Zhao, W.L. Yang and Y.Z. Zhang, *J. Stat. Mech.* 04 (2004) 04005
- [18] S.G. McAteer and M. Wheeler, *Nucl. Phys. B* 851 (2011) 346
- [19] M.J. Martins and M. Zuparic, *Nucl. Phys. B* 851 (2011) 565
- [20] O. Babelon, H.J. de Vega and C.M. Viallet, *Nucl. Phys. B* 190 (1981) 542
- [21] J.H.H Perk and C.L. Shultz, *Phys. Lett. A* 84 (1981) 407; C L Shultz, *Physica A* 122 (1983) 71
- [22] R.J. Baxter, *J.Stat.Phys.* 108 (2002) 1
- [23] R.A Pimenta and M.J. Martins, *J.Phys.A:Math.Theor.* 44 (2011) 085205
- [24] R. Hartshorne, *Algebraic Geometry* , Spring-Verlarg, New York (1977)
- [25] J. Schicho, *J.Symb.Comput.* 23 (1998) 191
- [26] A. Caradoc, O. Foda and N. Kitanine, *J.Stat.Mech.:Theory Exp.* P03012 (2006)
- [27] A. Down and O. Foda, *J.Stat.Mech.:Theory Exp.* P05010 (2006)

- [28] N.M. Bogoliubov, A.G. Pronko and M.B. Zvonarev, *J. Phys. A: Math. Gen.* 35 (2002) 5525
- [29] S.Y. Zhao, Y.Z. Zhang, *J.Math.Phys.* 48 (2007) 023504